Reasoning for Humans: Clear Thinking in an Uncertain World

PHIL 171

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Stochastic Truth Table

	P	Q	$\cdots$
p_1	T	T	
p_2	T	F	
p_3	F	T	
p_4	F	F	

A **Stochastic Truth Table** is a truth table where there is a number assigned to each row such that:

(1) each number is greater than or equal to 0; and

(2) and the sum of all the numbers assigned to the rows is 1

Given a stochastic truth table, how do you determine the probability of a formula X?

$Pr(X) =$ the sum of the probabilities assigned to rows that make X true.

prob	A	K	Q	
$\frac{1}{8}$	T	T	T	
$\frac{1}{8}$	T	T	F	
$\frac{1}{8}$	T	F	T	
$\frac{1}{8}$	T	F	F	
$\frac{1}{8}$	F	T	T	
$\frac{1}{8}$	F	T	F	
$\frac{1}{8}$	F	F	T	
$\frac{1}{8}$	F	F	F	

$$
\begin{aligned}
Pr(A) &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
&= \frac{4}{8} = \frac{1}{2}
\end{aligned}
$$

prob	A	K	Q	$A \wedge K$
$\frac{1}{8}$	T	T	T	T
$\frac{1}{8}$	T	T	F	T
$\frac{1}{8}$	T	F	T	F
$\frac{1}{8}$	T	F	F	F
$\frac{1}{8}$	F	T	T	F
$\frac{1}{8}$	F	T	F	F
$\frac{1}{8}$	F	F	T	F
$\frac{1}{8}$	F	F	F	F

$$\begin{aligned} Pr(A \wedge K) &= \frac{1}{8} + \frac{1}{8} \\ &= \frac{2}{8} = \frac{1}{4} \end{aligned}$$

prob	A	K	Q	$K \to Q$
$\frac{1}{8}$	T	T	T	T
$\frac{1}{8}$	T	T	F	F
$\frac{1}{8}$	T	F	T	T
$\frac{1}{8}$	T	F	F	T
$\frac{1}{8}$	F	T	T	T
$\frac{1}{8}$	F	T	F	F
$\frac{1}{8}$	F	F	T	T
$\frac{1}{8}$	F	F	F	T

$$
\begin{aligned}
Pr(K \to Q) &= 6 * \tfrac{1}{8} \\
&= \tfrac{6}{8} = \tfrac{3}{4}
\end{aligned}
$$

prob	A	K	Q	$K \to Q$
$\frac{1}{8}$	T	T	T	T
$\frac{1}{8}$	T	T	F	F
$\frac{1}{8}$	T	F	T	T
$\frac{1}{8}$	T	F	F	T
$\frac{1}{8}$	F	T	T	T
$\frac{1}{8}$	F	T	F	F
$\frac{1}{8}$	F	F	T	T
$\frac{1}{8}$	F	F	F	T

$$Pr(K \to Q) = 6 * \frac{1}{8}$$
$$= \frac{6}{8} = \frac{3}{4}$$

prob	A	K	Q	$K \to Q$
$\frac{1}{8}$	T	T	T	T
$\frac{1}{8}$	T	T	F	F
$\frac{1}{8}$	T	F	T	T
$\frac{1}{8}$	T	F	F	T
$\frac{1}{8}$	F	T	T	T
$\frac{1}{8}$	F	T	F	F
$\frac{1}{8}$	F	F	T	T
$\frac{1}{8}$	F	F	F	T

$$\begin{aligned} Pr(K \to Q) &= 6 * \tfrac{1}{8} \\ &= \tfrac{6}{8} = \tfrac{3}{4} \end{aligned}$$

prob	A	K	Q	
$\frac{1}{8}$	T	T	T	
$\frac{1}{8}$	T	T	F	
$\frac{1}{8}$	T	F	T	
$\frac{1}{8}$	T	F	F	
$\frac{1}{8}$	F	T	T	
$\frac{1}{8}$	F	T	F	
$\frac{1}{8}$	F	F	T	
$\frac{1}{8}$	F	F	F	

$$Pr(K \wedge \neg K) = 0$$

$$Pr((K \wedge Q) \to Q) = 1$$

$$Pr(\neg K \wedge \neg A) = \frac{2}{8} = \frac{1}{4}$$

$$Pr(\neg K \vee Q) = \frac{6}{8} = \frac{3}{4}$$

prob	A	K	Q	
$\frac{1}{8}$	T	T	T	
$\frac{1}{8}$	T	T	F	
$\frac{1}{8}$	T	F	T	
$\frac{1}{8}$	T	F	F	
$\frac{1}{8}$	F	T	T	
$\frac{1}{8}$	F	T	F	
$\frac{1}{8}$	F	F	T	
$\frac{1}{8}$	F	F	F	

$$Pr(K \wedge \neg K) = 0$$

$$Pr((K \wedge Q) \rightarrow Q) = 1$$

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$$Pr(\neg K \vee Q) = \frac{6}{8} = \frac{3}{4}$$

prob	A	K	Q	
$\frac{1}{8}$	T	T	T	
$\frac{1}{8}$	T	T	F	
$\frac{1}{8}$	T	F	T	
$\frac{1}{8}$	T	F	F	
$\frac{1}{8}$	F	T	T	
$\frac{1}{8}$	F	T	F	
$\frac{1}{8}$	F	F	T	
$\frac{1}{8}$	F	F	F	

$$Pr(K \wedge \neg K) = 0$$

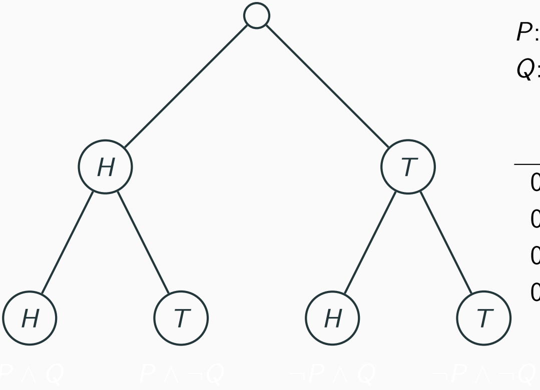
$$Pr((K \wedge Q) \rightarrow Q) = 1$$

$$Pr(\neg K \wedge \neg A) = \frac{2}{8} = \frac{1}{4}$$

$$Pr(\neg K \vee Q) = \frac{6}{8} = \frac{3}{4}$$

Suppose that you flip two fair coins. What is the probability that both land heads, both land tails, the first coin lands heads and the second lands tails, the first lands tails and the second lands heads?

Suppose that you flip two fair coins. What is the probability that both land heads, both land tails, the first coin lands heads and the second lands tails, the first lands tails and the second lands heads?

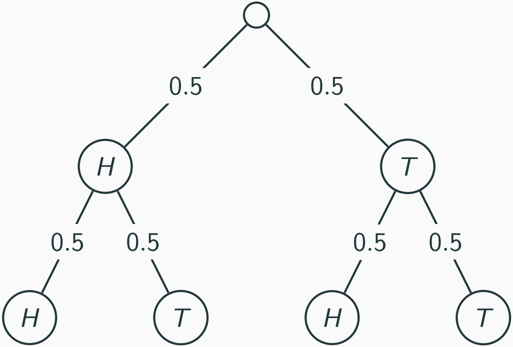


P: the first coin lands heads

Q: the second coin lands heads

	P	Q
0.25	T	T
0.25	T	F
0.25	F	T
0.25	F	F

Suppose that you flip two fair coins. What is the probability that both land heads, both land tails, the first coin lands heads and the second lands tails, the first lands tails and the second lands heads?

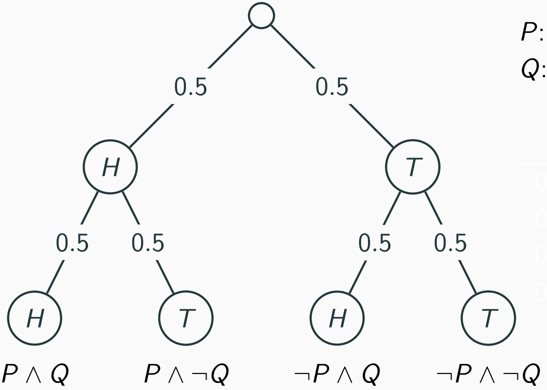


P: the first coin lands heads

Q: the second coin lands heads

	P	Q
0.25	T	T
0.25	T	F
0.25	F	T
0.25	F	F

Suppose that you flip two fair coins. What is the probability that both land heads, both land tails, the first coin lands heads and the second lands tails, the first lands tails and the second lands heads?

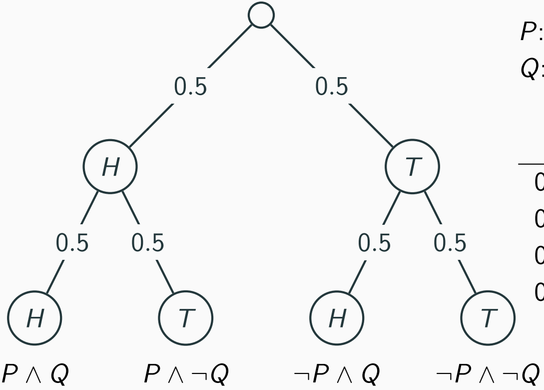


P: the first coin lands heads

Q: the second coin lands heads

	P	Q
0.25	T	T
0.25	T	F
0.25	F	T
0.25	F	F

Suppose that you flip two fair coins. What is the probability that both land heads, both land tails, the first coin lands heads and the second lands tails, the first lands tails and the second lands heads?

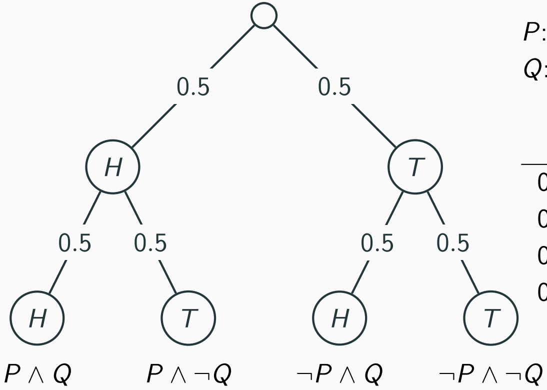


P: the first coin lands heads

Q: the second coin lands heads

	P	Q
$0.5 * 0.5$	T	T
$0.5 * 0.5$	T	F
$0.5 * 0.5$	F	T
$0.5 * 0.5$	F	F

Suppose that you flip two fair coins. What is the probability that both land heads, both land tails, the first coin lands heads and the second lands tails, the first lands tails and the second lands heads?

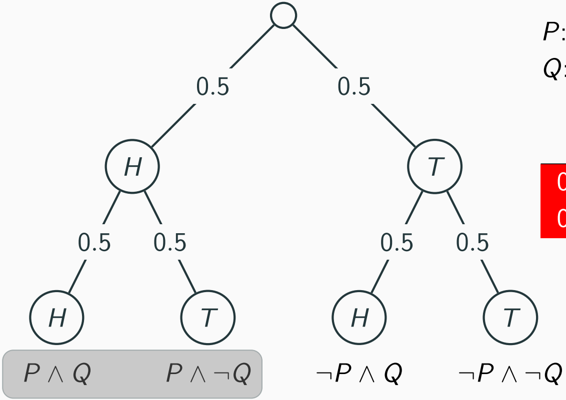


P: the first coin lands heads

Q: the second coin lands heads

	P	Q
0.25	T	T
0.25	T	F
0.25	F	T
0.25	F	F

Suppose that you flip two fair coins. **Suppose that you learn that the first coin landed heads.** What is the probability that the second coin landed tails?

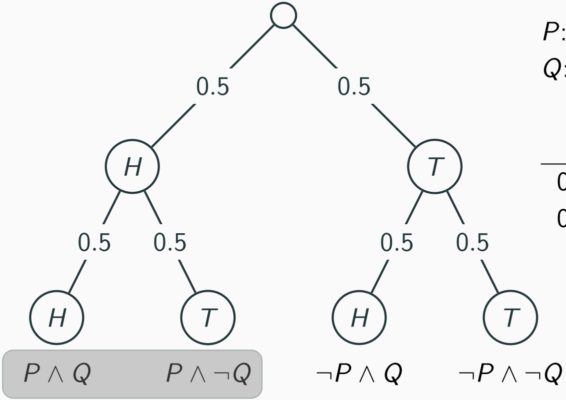


P: the first coin lands heads

Q: the second coin lands heads

	P	Q
0.25	T	T
0.25	T	F
0	F	T
0	F	F

Suppose that you flip two fair coins. **Suppose that you learn that the first coin landed heads.** What is the probability that the second coin landed tails? **Answer: 0.5**

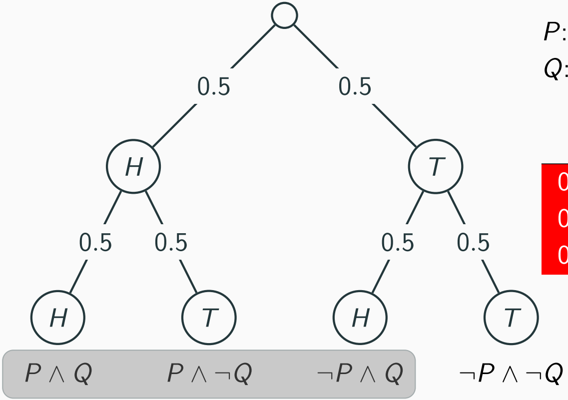


P: the first coin lands heads

Q: the second coin lands heads

	P	Q
0.5	T	T
0.5	T	F
0	F	T
0	F	F

Suppose that you flip two fair coins. **Suppose that you learn that at least one coin landed heads.** What is the probability that the second coin landed tails?

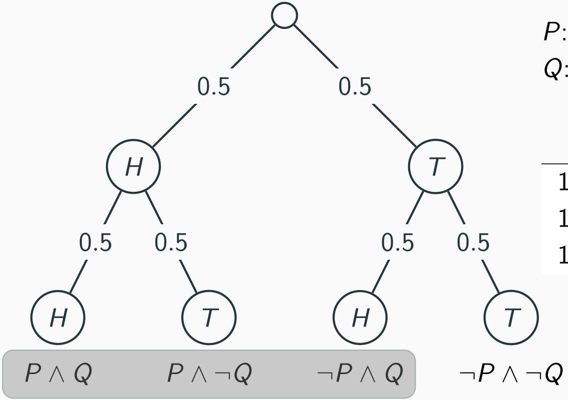


P: the first coin lands heads

Q: the second coin lands heads

	P	Q
0.5	T	T
0.5	T	F
0.5	F	T
0	F	F

Suppose that you flip two fair coins. **Suppose that you learn that at least one coin landed heads.** What is the probability that the second coin landed tails? **Answer: 1/3**



P: the first coin lands heads

Q: the second coin lands heads

	P	Q
1/3	T	T
1/3	T	F
1/3	F	T
0	F	F

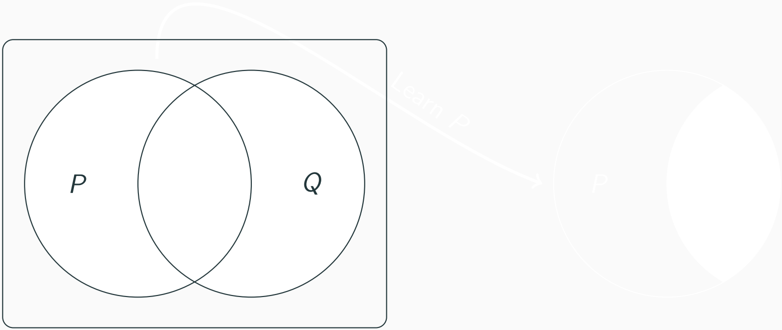
Conditional Probability

$Pr(X \mid Y)$: The probability of X **given that** Y.

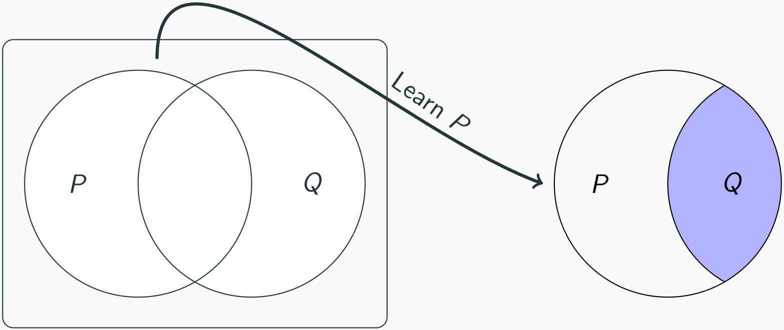
Conditional probability is the concept of the probability of something *given* or *in the light of* some evidence or new information.

Example:

- the probability that the die lands 1, given that it lands odd, is $1/3$
- the probability that it will rain tomorrow, given that there are dark clouds in the sky tomorrow morning, is high



$Pr(Q \mid P)$



$Pr(Q \mid P)$

Ratio Formula

$$Pr(X \mid Y) = \frac{Pr(X \wedge Y)}{Pr(Y)}$$

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$$Pr(X \mid Y) = \frac{Pr(X \wedge Y)}{Pr(Y)}$$

(If $Pr(Y) = 0$, then $Pr(X \mid Y)$ is undefined)

Example

	A	B
$\frac{1}{10}$	T	T
$\frac{1}{20}$	T	F
$\frac{2}{5}$	F	T
$\frac{9}{20}$	F	F

$$Pr(B) = \frac{1}{10} + \frac{2}{5} = \frac{5}{10}$$

Example

	A	B
$\frac{1}{10}$	T	T
$\frac{1}{20}$	T	F
$\frac{2}{5}$	F	T
$\frac{9}{20}$	F	F

$$Pr(B) = \frac{5}{10} \qquad Pr(B \mid A) = ??$$

Example

	A	B
$\frac{1}{10}$	T	T
$\frac{1}{20}$	T	F
0	F	T
0	F	F

$$Pr(B) = \frac{5}{10} \qquad Pr(B \mid A) = ??$$

Example

	A	B
$\frac{2}{3}$	T	T
$\frac{1}{3}$	T	F
0	F	T
0	F	F

$$Pr(B) = \frac{5}{10} \qquad Pr(B \mid A) = ??$$

Example

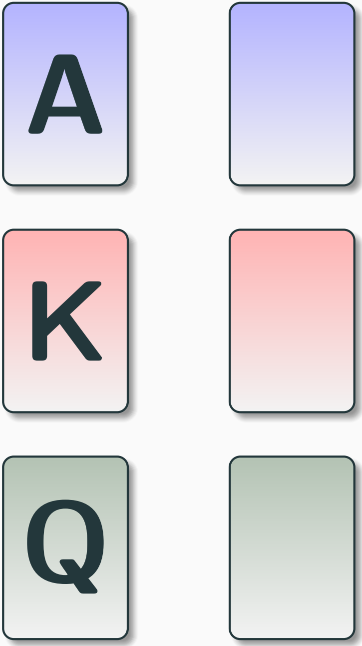
	A	B
$\frac{2}{3}$	T	T
$\frac{1}{3}$	T	F
0	F	T
0	F	F

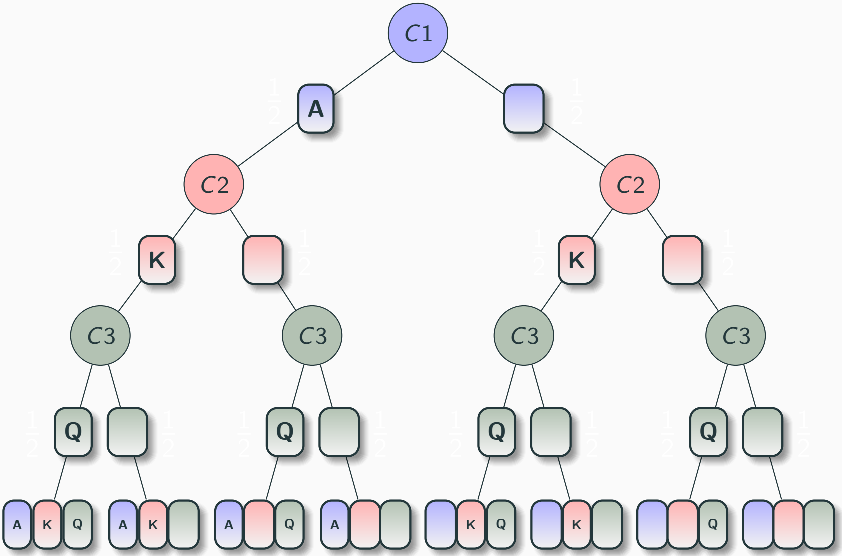
$$Pr(B) = \frac{5}{10} \qquad Pr(B \mid A) = \frac{2}{3}$$

Example

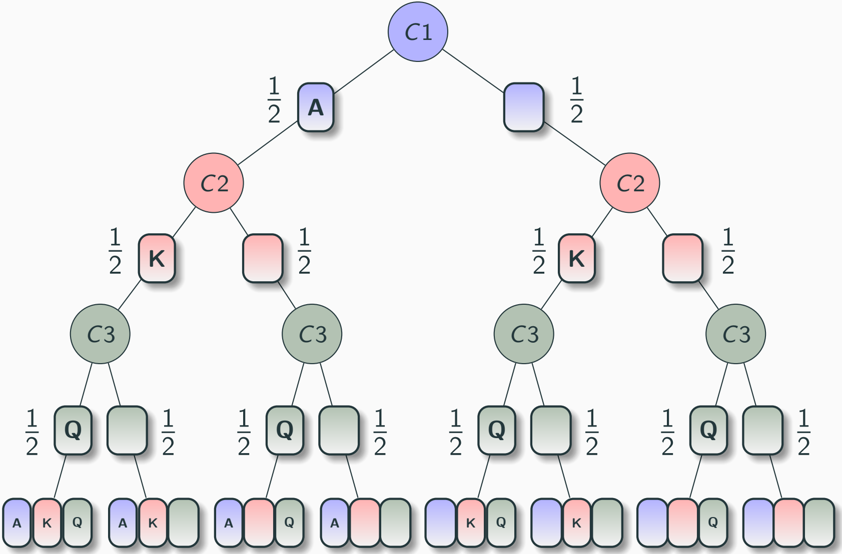
	A	B	$B \wedge A$
$\frac{1}{10}$	T	T	T
$\frac{1}{20}$	T	F	F
$\frac{2}{5}$	F	T	F
$\frac{9}{20}$	F	F	F

$$Pr(B) = \frac{5}{10} \qquad Pr(B \mid A) = \frac{Pr(B \wedge A)}{Pr(A)} = \frac{\frac{1}{10}}{\frac{1}{10} + \frac{1}{20}} = \frac{2}{3}$$

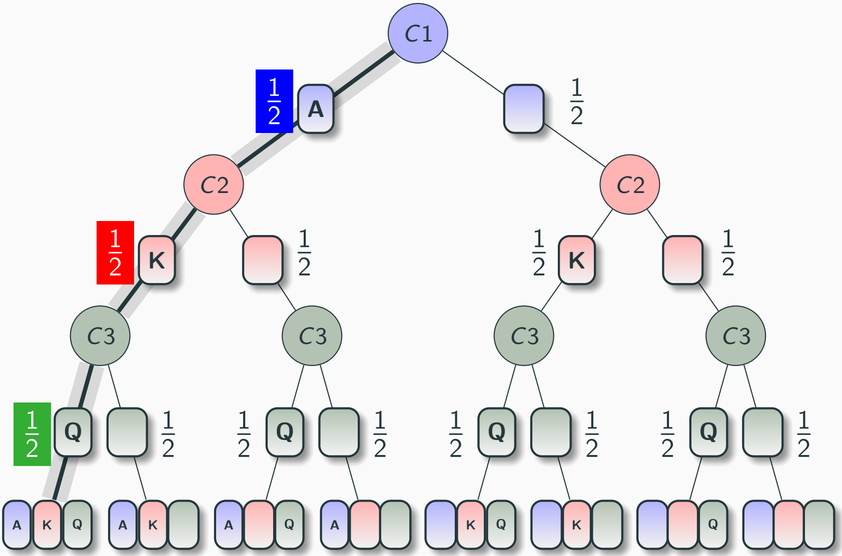

A
K
Q



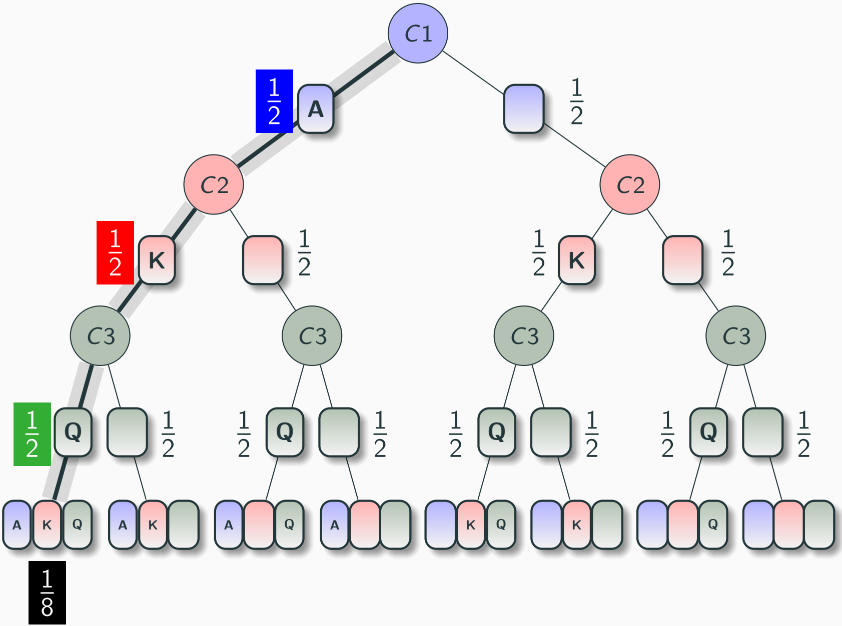
C1
A
C2
C2
K
K
C3
C3
C3
C3
Q
Q
Q
Q
A
K
Q
A
K
A
Q
A
K
Q
K
Q

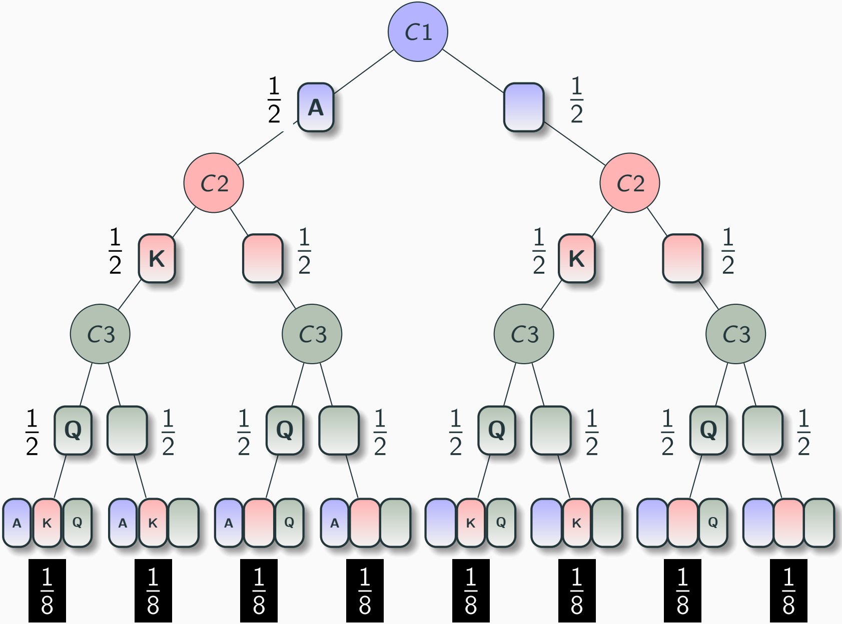


C1
1/2
A
1/2
C2
1/2
K
1/2
C2
1/2
K
1/2
C3
C3
C3
C3
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
A K Q
A K
A Q
A
K Q
K
Q



$$\frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8}$$


C1
1/2
A
1/2
C2
C2
1/2
K
1/2
1/2
K
1/2
C3
C3
C3
C3
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
A K Q
A K
A Q
A
K Q
K
Q
1/8


C1
1/2
A
1/2
C2
C2
1/2
K
1/2
1/2
K
1/2
C3
C3
C3
C3
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
A K Q
A K
A Q
A
K Q
K
Q
1/8
1/8
1/8
1/8
1/8
1/8
1/8
1/8

	A	K	Q	
	T	T	T	$A \wedge K \wedge Q$
	T	T	F	$A \wedge K \wedge \neg Q$
	T	F	T	$A \wedge \neg K \wedge Q$
	T	F	F	$A \wedge \neg K \wedge \neg Q$
	F	T	T	$\neg A \wedge K \wedge Q$
	F	T	F	$\neg A \wedge K \wedge \neg Q$
	F	F	T	$\neg A \wedge \neg K \wedge Q$
	F	F	F	$\neg A \wedge \neg K \wedge \neg Q$

prob	A	K	Q	
$\frac{1}{8}$	T	T	T	$A \wedge K \wedge Q$
$\frac{1}{8}$	T	T	F	$A \wedge K \wedge \neg Q$
$\frac{1}{8}$	T	F	T	$A \wedge \neg K \wedge Q$
$\frac{1}{8}$	T	F	F	$A \wedge \neg K \wedge \neg Q$
$\frac{1}{8}$	F	T	T	$\neg A \wedge K \wedge Q$
$\frac{1}{8}$	F	T	F	$\neg A \wedge K \wedge \neg Q$
$\frac{1}{8}$	F	F	T	$\neg A \wedge \neg K \wedge Q$
$\frac{1}{8}$	F	F	F	$\neg A \wedge \neg K \wedge \neg Q$

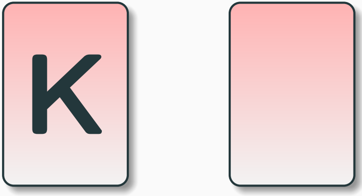
Biased dealer

- When selecting a red card, he is 3-times more likely to deal a K than a blank.

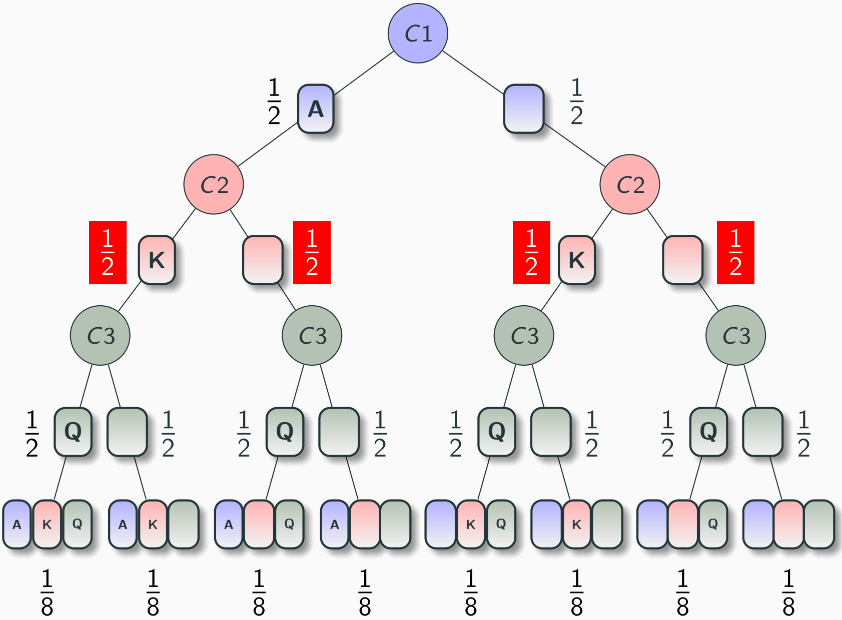
Biased dealer

- When selecting a red card, he is 3-times more likely to deal a K than a blank.

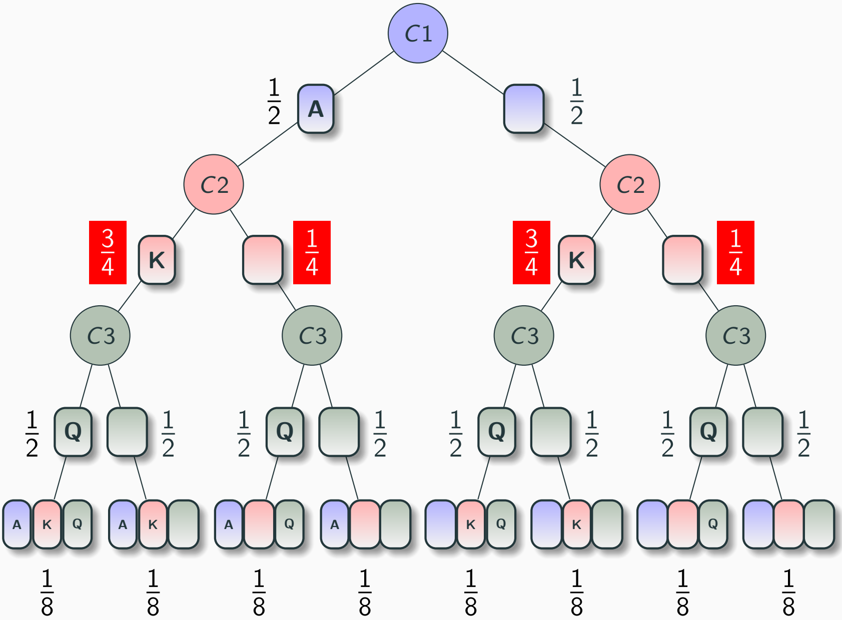
- When dealing the cards, he is 3-times more likely to deal cards so that the blue and red cards "match" (either both contain text or both are blank)



Suppose that the dealer is 3-times more likely to select red K than a blank red card.



C1
1/2
A
1/2
C2
C2
1/2
K
1/2
1/2
K
1/2
C3
C3
C3
C3
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
A K Q
A K
A Q
A
K Q
K
Q
1/8
1/8
1/8
1/8
1/8
1/8
1/8
1/8


C1
1/2
A
1/2
C2
C2
3/4
K
1/4
3/4
K
1/4
C3
C3
C3
C3
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
A K Q
A K
A Q
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K
Q
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1/8
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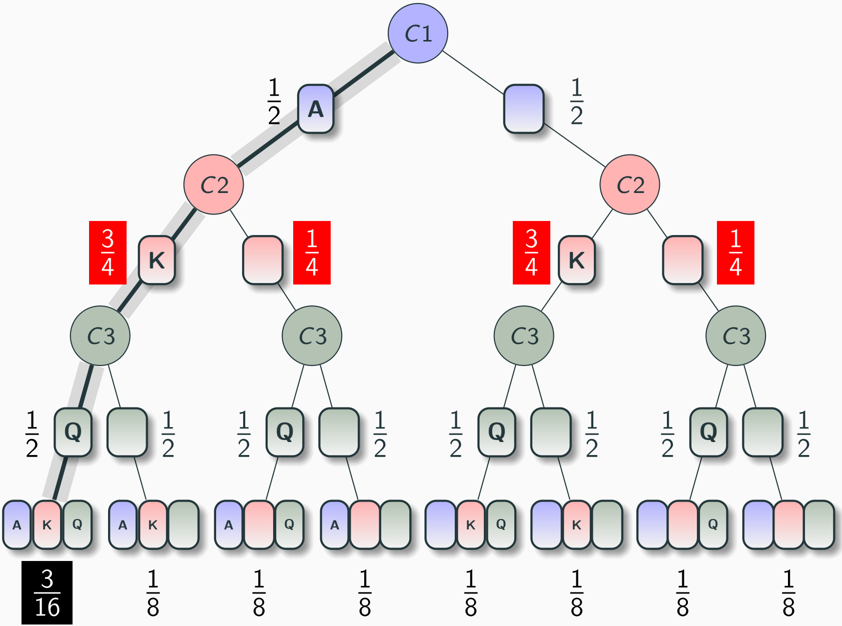
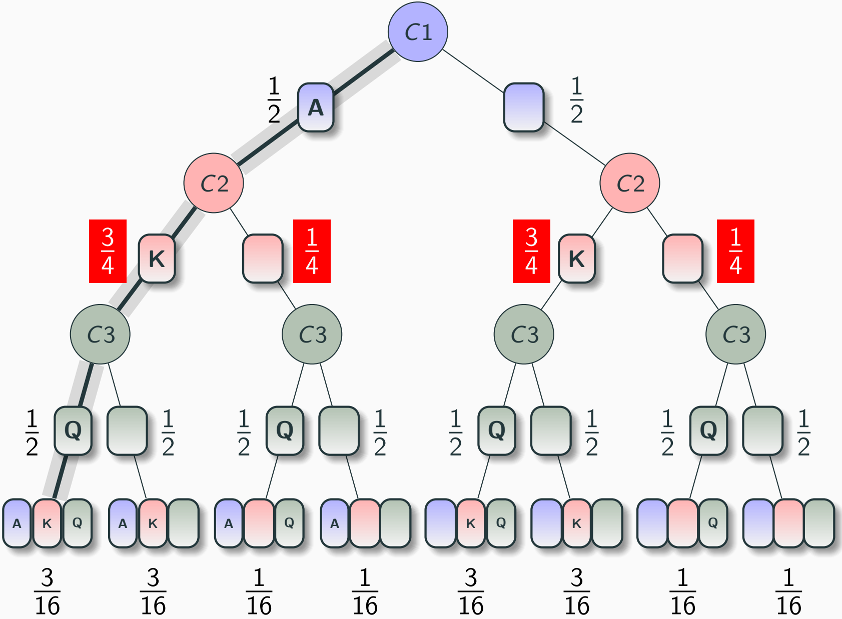
1/8
1/8
1/8



C1
1/2
A
1/2
C2
C2
3/4
K
1/4
3/4
K
1/4
C3
C3
C3
C3
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
A K Q
A K
A Q
A
K Q
K
Q
3/16
1/8
1/8
1/8
1/8
1/8
1/8
1/8



C1
1/2
A
1/2
C2
C2
3/4
K
1/4
3/4
K
1/4
C3
C3
C3
C3
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
A K Q
A K
A Q
A
K Q
K
Q
3/16
3/16
1/16
1/16
3/16
3/16
1/16
1/16

prob	A	K	Q	
$\frac{3}{16}$	T	T	T	
$\frac{3}{16}$	T	T	F	
$\frac{1}{16}$	T	F	T	
$\frac{1}{16}$	T	F	F	
$\frac{3}{16}$	F	T	T	
$\frac{3}{16}$	F	T	F	
$\frac{1}{16}$	F	F	T	
$\frac{1}{16}$	F	F	F	

$$Pr(A) =$$

$$Pr(K) =$$

$$Pr(A \wedge K) =$$

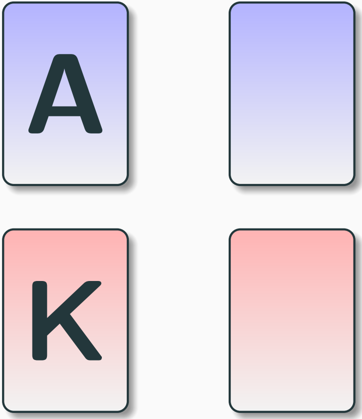
$$Pr(K \to A) =$$

$$Pr(A \mid K) =$$

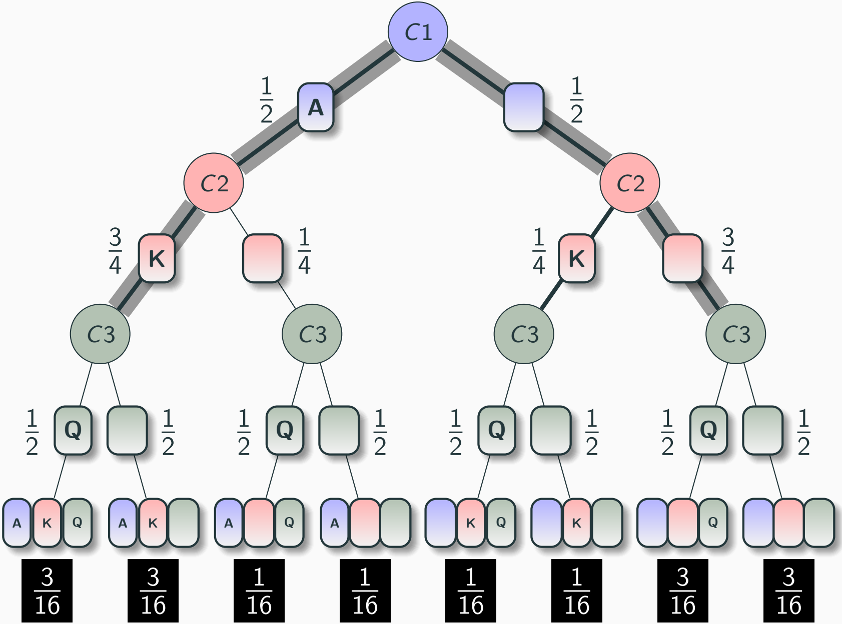
$$Pr(K \mid A) =$$

prob	A	K	Q	
$\frac{3}{16}$	T	T	T	
$\frac{3}{16}$	T	T	F	
$\frac{1}{16}$	T	F	T	
$\frac{1}{16}$	T	F	F	
$\frac{3}{16}$	F	T	T	
$\frac{3}{16}$	F	T	F	
$\frac{1}{16}$	F	F	T	
$\frac{1}{16}$	F	F	F	

$$
\begin{aligned}
Pr(A) &= \tfrac{8}{16} = \tfrac{1}{2} \\
Pr(K) &= \tfrac{12}{16} = \tfrac{3}{4} \\
Pr(A \wedge K) &= \tfrac{6}{16} \\
Pr(K \rightarrow A) &= \tfrac{10}{16} \\
Pr(A \mid K) &= \tfrac{Pr(A \wedge K)}{Pr(K)} = \tfrac{\frac{6}{16}}{\frac{12}{16}} = \tfrac{1}{2} \\
Pr(K \mid A) &= \tfrac{Pr(K \wedge A)}{Pr(A)} = \tfrac{\frac{6}{16}}{\frac{8}{16}} = \tfrac{3}{4}
\end{aligned}
$$



Suppose that the dealer is 3-times more likely to deal cards so that the blue and red cards “match” (either both contain text or both are blank).


C1
1/2
A
1/2
C2
C2
3/4
K
1/4
1/4
K
3/4
C3
C3
C3
C3
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
1/2
Q
1/2
A K Q
A K
A Q
A
K Q
K
Q
3/16
3/16
1/16
1/16
1/16
1/16
3/16
3/16

prob	A	K	Q	
$\frac{3}{16}$	T	T	T	
$\frac{3}{16}$	T	T	F	
$\frac{1}{16}$	T	F	T	
$\frac{1}{16}$	T	F	F	
$\frac{1}{16}$	F	T	T	
$\frac{1}{16}$	F	T	F	
$\frac{3}{16}$	F	F	T	
$\frac{3}{16}$	F	F	F	

$$Pr(A) =$$

$$Pr(K) =$$

$$Pr(A \wedge K) =$$

$$Pr(A \mid K) =$$

$$Pr(K \mid A) =$$

$$Pr(Q \mid A) =$$

prob	A	K	Q	
$\frac{3}{16}$	T	T	T	
$\frac{3}{16}$	T	T	F	
$\frac{1}{16}$	T	F	T	
$\frac{1}{16}$	T	F	F	
$\frac{1}{16}$	F	T	T	
$\frac{1}{16}$	F	T	F	
$\frac{3}{16}$	F	F	T	
$\frac{3}{16}$	F	F	F	

$$
\begin{aligned}
Pr(A) &= \frac{8}{16} = \frac{1}{2} \\
Pr(K) &= \frac{8}{16} = \frac{1}{2} \\
Pr(A \wedge K) &= \frac{6}{16} = \frac{3}{8} \\
Pr(A \mid K) &= \frac{Pr(A \wedge K)}{Pr(K)} = \frac{\frac{6}{16}}{\frac{8}{16}} = \frac{3}{4} \\
Pr(K \mid A) &= \frac{Pr(K \wedge A)}{Pr(A)} = \frac{\frac{6}{16}}{\frac{8}{16}} = \frac{3}{4} \\
Pr(Q \mid A) &= \frac{Pr(Q \wedge A)}{Pr(A)} = \frac{\frac{4}{16}}{\frac{8}{16}} = \frac{1}{2}
\end{aligned}
$$

Kolmogrov Axioms

Given any stochastic truth table, for any formulas X and Y:

- $Pr(X) \geq 0$
- If X is a tautology, then $Pr(X) = 1$
- If X and Y are mutually exclusive, then $Pr(X \vee Y) = Pr(X) + Pr(Y)$

Laws of Probability

In any stochastic truth table, for all X, $Pr(\neg X) = 1 - Pr(X)$

In any stochastic truth table, for all X, if X is a contradiction, then $Pr(X) = 0$

In any stochastic truth table, for all X and Y, if $X \leftrightarrow Y$ is a tautology (i.e., X and Y are tautologically equivalent), then $Pr(X) = Pr(Y)$

Laws of Probability

In any stochastic truth table, for all X and Y,
$Pr(X) = Pr(X \wedge Y) + Pr(X \wedge \neg Y)$

In any stochastic truth table, for all X and Y,
$Pr(X \vee Y) = Pr(X) + Pr(Y) - Pr(X \wedge Y)$

In any stochastic truth table, for all X and Y,
if $X \rightarrow Y$ is a tautology, then $Pr(X) \leq Pr(Y)$

Laws of Total Probability

In any stochastic truth table, for all X and Y,

$$Pr(X) = Pr(Y)Pr(X \mid Y) + Pr(\neg Y)Pr(X \mid \neg Y)$$