Reasoning for Humans: Clear Thinking in an Uncertain World

PHIL 171

Eric Pacuit

Department of Philosophy
University of Maryland
pacuit.org

Proportions/Probability

In a weighted truth table, the proportion or probability, that φ is true is:

$$Pr(\varphi) = \frac{\#(\varphi)}{\#(\top)}$$

	I	P
80	T	T
20	T	F
100	F	T
900	F	F

$$
\begin{aligned}
\#(P \wedge I) &= 80 & Pr(P \wedge I) &= \tfrac{80}{1100} = \tfrac{4}{55} \\
\#(P) &= 180 & Pr(P) &= \tfrac{180}{1100} = \tfrac{9}{55} \\
\#(I) &= 100 & Pr(I) &= \tfrac{100}{1100} = \tfrac{1}{11} \\
\#(\neg P) &= 1000 & Pr(\neg P) &= \tfrac{1000}{1100} = \tfrac{10}{11} \\
\#(P \vee I) &= 200 & Pr(P \vee I) &= \tfrac{200}{1100} = \tfrac{2}{11} \\
\#(P \vee \neg P) &= 1100 & Pr(P \vee \neg P) &= \tfrac{1100}{1100} = 1
\end{aligned}
$$

Truth Table

	P	Q	$\cdots$
1.	T	T	
2.	T	F	
3.	F	T	
4.	F	F	

Stochastic Truth Table

	P	Q	$\cdots$
p_1	T	T	
p_2	T	F	
p_3	F	T	
p_4	F	F	

A **Stochastic Truth Table** is a truth table where there is a number assigned to each row such that:

(1) each number is greater than or equal to 0; and

(2) and the sum of all the numbers assigned to the rows is 1

Stochastic Truth Table

	P	Q	$\cdots$
p_1	T	T	
p_2	T	F	
p_3	F	T	
p_4	F	F	

A **Stochastic Truth Table** is a truth table where each row i is assigned a number p_i such that:

(1) For each i, $p_i \geq 0$; and

(2) $\sum_i p_i = 1$

Stochastic Truth Table

	P	Q	$\cdots$
p_1	T	T	
p_2	T	F	
p_3	F	T	
p_4	F	F	

p_i is a measure of how "likely" it is that the situation described by row i obtains ("the *probability* of row i")

For example, p_2 is the measure of how likely it is that P is true and Q is false.

Stochastic Truth Table

	P	Q	$\cdots$
$\frac{2}{3}$	T	T	
$\frac{1}{6}$	T	F	
$\frac{1}{12}$	F	T	
$\frac{1}{12}$	F	F	

$\frac{2}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = 1$

The probability that P is true and Q is true is $\frac{2}{3}$.

The probability that P is true and Q is false is $\frac{1}{6}$.

The probability that P is false and Q is true is $\frac{1}{12}$.

The probability that P is false and Q is false is $\frac{1}{12}$.

Stochastic Truth Table

	P	Q	$\cdots$
$\frac{1}{4}$	T	T	
$\frac{1}{4}$	T	F	
$\frac{1}{4}$	F	T	
$\frac{1}{4}$	F	F	

$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$

The probability that P is true and Q is true is $\frac{1}{4}$.

The probability that P is true and Q is false is $\frac{1}{4}$.

The probability that P is false and Q is true is $\frac{1}{4}$.

The probability that P is false and Q is false is $\frac{1}{4}$.

Stochastic Truth Table

	P	Q	$\cdots$
$\frac{1}{4}$	T	T	
0	T	F	
0	F	T	
$\frac{3}{4}$	F	F	

$\frac{1}{4} + 0 + 0 + \frac{3}{4} = 1$

The probability that P is true and Q is true is $\frac{1}{4}$.

The probability that P is true and Q is false is 0.

The probability that P is false and Q is true is 0.

The probability that P is false and Q is false is $\frac{3}{4}$.

- *Any* assignment of nonnegative numbers that sum to 1 can be used to form a stochastic truth table.

- *Any* assignment of nonnegative numbers that sum to 1 can be used to form a stochastic truth table.

- Since the sum must be 1 and the numbers assigned to a row in a stochastic truth table are all greater than or equal to 0, no row is assigned a number greater than 1.

- *Any* assignment of nonnegative numbers that sum to 1 can be used to form a stochastic truth table.

- Since the sum must be 1 and the numbers assigned to a row in a stochastic truth table are all greater than or equal to 0, no row is assigned a number greater than 1.

- If a row is assigned 1, then that row is certain to obtain and all other rows must be assigned 0.

- *Any* assignment of nonnegative numbers that sum to 1 can be used to form a stochastic truth table.

- Since the sum must be 1 and the numbers assigned to a row in a stochastic truth table are all greater than or equal to 0, no row is assigned a number greater than 1.

- If a row is assigned 1, then that row is certain to obtain and all other rows must be assigned 0.

- If a row is assigned 0, then it is a logical possibility that certainly will not happen (e.g., being dealt a red spade or flipping a coin and it landing on its side).

Given a stochastic truth table, how do you determine the probability of a formula φ?

$$Pr(\varphi) = \sum\{p_i \mid \ i \text{ is a row that makes } \varphi \text{ true}\}$$

prob	A	K	Q	
$\frac{1}{8}$	T	T	T	
$\frac{1}{8}$	T	T	F	
$\frac{1}{8}$	T	F	T	
$\frac{1}{8}$	T	F	F	
$\frac{1}{8}$	F	T	T	
$\frac{1}{8}$	F	T	F	
$\frac{1}{8}$	F	F	T	
$\frac{1}{8}$	F	F	F	

$$
\begin{aligned}
Pr(A) &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
&= \frac{4}{8} = \frac{1}{2}
\end{aligned}
$$

prob	A	K	Q	$A \wedge K$
$\frac{1}{8}$	T	T	T	T
$\frac{1}{8}$	T	T	F	T
$\frac{1}{8}$	T	F	T	F
$\frac{1}{8}$	T	F	F	F
$\frac{1}{8}$	F	T	T	F
$\frac{1}{8}$	F	T	F	F
$\frac{1}{8}$	F	F	T	F
$\frac{1}{8}$	F	F	F	F

$$
\begin{aligned}
Pr(A \wedge K) &= \frac{1}{8} + \frac{1}{8} \\
&= \frac{2}{8} = \frac{1}{4}
\end{aligned}
$$

prob	A	K	Q	$K \to Q$
$\frac{1}{8}$	T	T	T	T
$\frac{1}{8}$	T	T	F	F
$\frac{1}{8}$	T	F	T	T
$\frac{1}{8}$	T	F	F	T
$\frac{1}{8}$	F	T	T	T
$\frac{1}{8}$	F	T	F	F
$\frac{1}{8}$	F	F	T	T
$\frac{1}{8}$	F	F	F	T

$$Pr(K \to Q) = 6 * \frac{1}{8}$$

$$= \frac{6}{8} = \frac{3}{4}$$

prob	A	K	Q	$K \rightarrow Q$
$\frac{1}{8}$	T	T	T	T
$\frac{1}{8}$	T	T	F	F
$\frac{1}{8}$	T	F	T	T
$\frac{1}{8}$	T	F	F	T
$\frac{1}{8}$	F	T	T	T
$\frac{1}{8}$	F	T	F	F
$\frac{1}{8}$	F	F	T	T
$\frac{1}{8}$	F	F	F	T

$$Pr(K \rightarrow Q) = 6 * \frac{1}{8}$$
$$= \frac{6}{8} = \frac{3}{4}$$

prob	A	K	Q	$K \to Q$
$\frac{1}{8}$	T	T	T	T
$\frac{1}{8}$	T	T	F	F
$\frac{1}{8}$	T	F	T	T
$\frac{1}{8}$	T	F	F	T
$\frac{1}{8}$	F	T	T	T
$\frac{1}{8}$	F	T	F	F
$\frac{1}{8}$	F	F	T	T
$\frac{1}{8}$	F	F	F	T

$$
\begin{aligned}
Pr(K \to Q) &= 6 * \frac{1}{8} \\
&= \frac{6}{8} = \frac{3}{4}
\end{aligned}
$$

prob	A	K	Q	
$\frac{1}{8}$	T	T	T	
$\frac{1}{8}$	T	T	F	
$\frac{1}{8}$	T	F	T	
$\frac{1}{8}$	T	F	F	
$\frac{1}{8}$	F	T	T	
$\frac{1}{8}$	F	T	F	
$\frac{1}{8}$	F	F	T	
$\frac{1}{8}$	F	F	F	

$$Pr(K \wedge \neg K) = 0$$

$$Pr((K \wedge Q) \to Q) = 1$$

$$Pr(\neg K \wedge \neg A) = \frac{2}{8} = \frac{1}{4}$$

$$Pr(\neg K \vee Q) = \frac{6}{8} = \frac{3}{4}$$

prob	A	K	Q	
$\frac{1}{8}$	T	T	T	
$\frac{1}{8}$	T	T	F	
$\frac{1}{8}$	T	F	T	
$\frac{1}{8}$	T	F	F	
$\frac{1}{8}$	F	T	T	
$\frac{1}{8}$	F	T	F	
$\frac{1}{8}$	F	F	T	
$\frac{1}{8}$	F	F	F	

$$Pr(K \wedge \neg K) = 0$$

$$Pr((K \wedge Q) \rightarrow Q) = 1$$

$$Pr(\neg K \wedge \neg A) = \frac{2}{8} = \frac{1}{4}$$

$$Pr(\neg K \vee Q) = \frac{6}{8} = \frac{3}{4}$$

prob	A	K	Q	
$\frac{1}{8}$	T	T	T	
$\frac{1}{8}$	T	T	F	
$\frac{1}{8}$	T	F	T	
$\frac{1}{8}$	T	F	F	
$\frac{1}{8}$	F	T	T	
$\frac{1}{8}$	F	T	F	
$\frac{1}{8}$	F	F	T	
$\frac{1}{8}$	F	F	F	

$$Pr(K \wedge \neg K) = 0$$

$$Pr((K \wedge Q) \to Q) = 1$$

$$Pr(\neg K \wedge \neg A) = \frac{2}{8} = \frac{1}{4}$$

$$Pr(\neg K \vee Q) = \frac{6}{8} = \frac{3}{4}$$

Conditional Probability

In a weighted truth table, we are often interested in what proportion of the trials that make ψ true also make φ true?

$$Pr(\varphi \mid \psi) = \frac{\#(\varphi \land \psi)}{\#(\psi)}$$

Conditional Probability

In a weighted truth table, we are often interested in what proportion of the trials that make ψ true also make φ true?

$$Pr(\varphi \mid \psi) = \frac{\#(\varphi \wedge \psi)}{\#(\psi)}$$

	I	P
80	T	T
20	T	F
100	F	T
900	F	F

$$Pr(I \mid P) = \frac{\#(I \wedge P)}{\#(P)} = \frac{80}{180} = \frac{4}{9}$$

	I	P
80	T	T
20	T	F
100	F	T
900	F	F

$$
\begin{aligned}
Pr(I \mid P) &= \tfrac{\#(I \wedge P)}{\#(P)} = \tfrac{80}{180} = \tfrac{4}{9} \\
Pr(I \mid \neg P) &= \\
Pr(P \mid I) &= \\
Pr(P \mid \neg I) &=
\end{aligned}
$$

	I	P
80	T	T
20	T	F
100	F	T
900	F	F

$$
\begin{aligned}
Pr(I \mid P) &= \tfrac{\#(I \wedge P)}{\#(P)} = \tfrac{80}{180} = \tfrac{4}{9} \\
Pr(I \mid \neg P) &= \tfrac{\#(I \wedge \neg P)}{\#(\neg P)} = \tfrac{20}{920} = \tfrac{1}{46} \\
Pr(P \mid I) &= \tfrac{\#(P \wedge I)}{\#(I)} = \tfrac{80}{100} = \tfrac{4}{5} \\
Pr(P \mid \neg I) &= \tfrac{\#(P \wedge \neg I)}{\#(\neg I)} = \tfrac{100}{1000} = \tfrac{1}{10}
\end{aligned}
$$

Conditional Probability

In a stochastic truth table, the probability of φ conditional on ψ is:

$$Pr(\varphi \mid \psi) = \frac{Pr(\varphi \wedge \psi)}{Pr(\psi)}$$

prob	A	K	Q	
$\frac{1}{8}$	T	T	T	
$\frac{1}{8}$	T	T	F	
$\frac{1}{8}$	T	F	T	
$\frac{1}{8}$	T	F	F	
$\frac{1}{8}$	F	T	T	
$\frac{1}{8}$	F	T	F	
$\frac{1}{8}$	F	F	T	
$\frac{1}{8}$	F	F	F	

$$Pr(K \mid Q) = \frac{Pr(K \wedge Q)}{Pr(Q)} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$Pr(K) = \frac{1}{2}$$

$$Pr(Q \to K) = \frac{3}{4}$$

In any stochastic truth table,

$$Pr(\neg\varphi) = 1 - Pr(\varphi)$$

$$Pr(\varphi \vee \psi) = Pr(\varphi) + Pr(\psi) - Pr(\varphi \wedge \psi)$$

$$Pr(\varphi \wedge \psi) = Pr(\varphi \mid \psi)Pr(\psi)$$

Example

- **four-fifths** of all Samson wheelchairs are built in Boston;
- the rest are built in Chicago;
- **one-sixth** of the wheelchairs built in Boston have a special aluminum frame;
- **three-quarters** of the wheelchairs built in Chicago have the aluminum frame.

- **four-fifths** of all Samson wheelchairs are built in Boston;
- the rest are built in Chicago;
- **one-sixth** of the wheelchairs built in Boston have a special aluminum frame;
- **three-quarters** of the wheelchairs built in Chicago have the aluminum frame.

A: The wheelchair has a special aluminum frame.
B: The wheelchair was built in Boston.

	A	B	
p_1	T	T	
p_2	T	F	
p_3	F	T	
p_4	F	F	

- **four-fifths** of all Samson wheelchairs are built in Boston;
- the rest are built in Chicago;
- **one-sixth** of the wheelchairs built in Boston have a special aluminum frame;
- **three-quarters** of the wheelchairs built in Chicago have the aluminum frame.

A: The wheelchair has a special aluminum frame.
B: The wheelchair was built in Boston.

	A	B	
p_1	T	T	$A \wedge B$
p_2	T	F	$A \wedge \neg B$
p_3	F	T	$\neg A \wedge B$
p_4	F	F	$\neg A \wedge \neg B$

- **four-fifths** of all Samson wheelchairs are built in Boston;
- the rest are built in Chicago;
- **one-sixth** of the wheelchairs built in Boston have a special aluminum frame;
- **three-quarters** of the wheelchairs built in Chicago have the aluminum frame.

A: The wheelchair has a special aluminum frame.
B: The wheelchair was built in Boston.

$$\begin{aligned} Pr(A \wedge B) &= \\ Pr(A \wedge \neg B) &= \\ Pr(\neg A \wedge B) &= \\ Pr(\neg A \wedge \neg B) &= \end{aligned}$$

- **four-fifths** of all Samson wheelchairs are built in Boston;
- the rest are built in Chicago;
- **one-sixth** of the wheelchairs built in Boston have a special aluminum frame;
- **three-quarters** of the wheelchairs built in Chicago have the aluminum frame.

A: The wheelchair has a special aluminum frame.
B: The wheelchair was built in Boston.

$$\begin{aligned}
Pr(A \wedge B) &= Pr(A \mid B)Pr(B) &&= 1/6 \times 4/5 = 4/30 \\
Pr(A \wedge \neg B) &= Pr(A \mid \neg B)Pr(\neg B) &&= 3/4 \times 1/5 = 3/20 \\
Pr(\neg A \wedge B) &= Pr(\neg A \mid B)Pr(B) &&= 5/6 \times 4/5 = 20/30 \\
Pr(\neg A \wedge \neg B) &= Pr(\neg A \mid \neg B)Pr(\neg B) &&= 1/4 \times 1/5 = 1/20
\end{aligned}$$

- **four-fifths** of all Samson wheelchairs are built in Boston;
- the rest are built in Chicago;
- **one-sixth** of the wheelchairs built in Boston have a special aluminum frame;
- **three-quarters** of the wheelchairs built in Chicago have the aluminum frame.

A: The wheelchair has a special aluminum frame.

B: The wheelchair was built in Boston.

	A	B	
4/30	T	T	$A \wedge B$
3/20	T	F	$A \wedge \neg B$
20/30	F	T	$\neg A \wedge B$
1/20	F	F	$\neg A \wedge \neg B$