Reasoning for Humans: Clear Thinking in an Uncertain World

PHIL 171

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Probability/Inductive Logic

An argument is **deductively valid** if and only if it is *impossible* that its conclusion is false while its premises are true.

An argument is **inductively strong** if and only if it is *improbable* that its conclusion is false while its premises are true.

Example

1,100 people, who are known (by some definitive means) to either be infected or not infected by the coronavirus, were given an inexpensive test to see how well it works.

I means the person is infected

P means the person tests positive

I	P
T	T
T	F
F	T
F	F

Suppose the 1,100 people were drawn at random from the general population. Now you take the test and test positive. How confident are you that you're infected?

Example

1,100 people, who are known (by some definitive means) to either be infected or not infected by the coronavirus, were given an inexpensive test to see how well it works.

I means the person is infected

P means the person tests positive

	I	P
80	T	T
20	T	F
100	F	T
900	F	F

Suppose the 1,100 people were drawn at random from the general population. Now you take the test and test positive. How confident are you that you're infected?

Weighted Truth Table

A **weighted truth table** is a truth table where each row is assigned a nonnegative number (an integer greater than or equal to 0).

For a formula of propositional logic φ, let $\#(\varphi)$ be the number of ways that φ is true. I.e., it is the sum of the numbers assigned to the rows that make φ true.

Example

	I	P
80	T	T
20	T	F
100	F	T
900	F	F

$$\begin{aligned}
\#(P \wedge I) &= \\
\#(P) &= \\
\#(I) &= \\
\#(\neg P) &= \\
\#(P \vee I) &= \\
\#(P \vee \neg P) &=
\end{aligned}$$

Example

	I	P
80	T	T
20	T	F
100	F	T
900	F	F

$$\begin{aligned}
\#(P \wedge I) &= 80 \\
\#(P) &= \\
\#(I) &= \\
\#(\neg P) &= \\
\#(P \vee I) &= \\
\#(P \vee \neg P) &=
\end{aligned}$$

Example

	I	P
80	T	T
20	T	F
100	F	T
900	F	F

$$
\begin{aligned}
\#(P \wedge I) &= 80 \\
\#(P) &= 80 + 100 = 180 \\
\#(I) &= 80 + 20 = 100 \\
\#(\neg P) &= 20 + 900 = 920 \\
\#(P \vee I) &= 80 + 20 + 100 = 200 \\
\#(P \vee \neg P) &= 80 + 20 + 100 + 900 = 1100
\end{aligned}
$$

Example

	I	P
0	T	T
100	T	F
180	F	T
820	F	F

$$
\begin{aligned}
\#(P \wedge I) &= 0 \\
\#(P) &= 0 + 180 = 180 \\
\#(I) &= 0 + 100 = 100 \\
\#(\neg P) &= 100 + 820 = 920 \\
\#(P \vee I) &= 0 + 100 + 180 = 280 \\
\#(P \vee \neg P) &= 0 + 100 + 180 + 820 = 1100
\end{aligned}
$$

Let $\top$ be a special formula that is always true. Then, in any weighted truth-table, $\#(\top)$ is the sum of the numbers assigned to the rows.

In any weighted truth table, for any formula φ,

$$\#(\neg\varphi) = \#(\top) - \#(\varphi)$$

A very important point

$\#_1$	I	P
80	T	T
20	T	F
100	F	T
900	F	F

$\#_2$	I	P
0	T	T
100	T	F
180	F	T
820	F	F

$$\#_1(P) = \#_2(P) = 180$$

$$\#_1(I) = \#_2(I) = 100$$

$$\#_1(\top) = \#_2(\top) = 1100$$

A very important point

$\#_1$	I	P
80	T	T
20	T	F
100	F	T
900	F	F

$\#_2$	I	P
0	T	T
100	T	F
180	F	T
820	F	F

$$\#_1(P) = \#_2(P) = 180$$

$$\#_1(I) = \#_2(I) = 100$$

$$\#_1(\top) = \#_2(\top) = 1100$$

$$1100 - 180 = 920 = \#_1(\neg P) = \#_2(\neg P) = 920 = 1100 - 180$$

A very important point

$\#_1$	I	P
80	T	T
20	T	F
100	F	T
900	F	F

$\#_2$	I	P
0	T	T
100	T	F
180	F	T
820	F	F

$$\#_1(P) = \#_2(P) = 180$$

$$\#_1(I) = \#_2(I) = 100$$

$$\#_1(\top) = \#_2(\top) = 1100$$

$$200 = \#_1(P \vee I) \neq \#_2(P \vee I) = 280$$

In any weighted truth table, for all formulas φ and ψ,

$$\text{if } \#(\varphi \wedge \psi) = 0, \text{ then } \#(\varphi \vee \psi) = \#(\varphi) + \#(\psi)$$

In any weighted truth table, for all formulas φ and ψ,

$$\text{if } \#(\varphi \wedge \psi) = 0, \text{ then } \#(\varphi \vee \psi) = \#(\varphi) + \#(\psi)$$

In any weighted truth table, for all formulas φ and ψ,

$$\#(\varphi \vee \psi) = \#(\varphi) + \#(\psi) - \#(\varphi \wedge \psi)$$

If φ is a contradiction, then in any weighted truth table, $\#(\varphi) = 0$

If φ is a tautology, then in any weighted truth table, $\#(\varphi) = \#(\top)$

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If φ is a tautology, then in any weighted truth table, $\#(\varphi) = \#(\top)$

If φ and ψ are mutually exclusive, then in any weighted truth table, $\#(\varphi \wedge \psi) = 0$

If φ is a contradiction, then in any weighted truth table, $\#(\varphi) = 0$

If φ is a tautology, then in any weighted truth table, $\#(\varphi) = \#(\top)$

If φ and ψ are mutually exclusive, then in any weighted truth table, $\#(\varphi \wedge \psi) = 0$

If φ and ψ are tautologically equivalent, then in any weighted truth table, $\#(\varphi) = \#(\psi)$

For all formulas φ and ψ, then in any weighted truth table, $\#(\varphi) = \#(\varphi \wedge \psi) + \#(\varphi \wedge \neg\psi)$.

For all formulas φ and ψ, then in any weighted truth table, $\#(\varphi) = \#(\varphi \wedge \psi) + \#(\varphi \wedge \neg\psi)$.

Since $\varphi \approx (\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi)$, $\#(\varphi) = \#((\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi))$

Since $\#((\varphi \wedge \psi) \wedge (\varphi \wedge \neg\psi)) = 0$, $\#((\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi)) = \#(\varphi \wedge \psi) + \#(\varphi \wedge \neg\psi)$

If $\varphi \to \psi$ is a tautology, then in any weighted truth table, $\#(\varphi) \leq \#(\psi)$.

If $\varphi \to \psi$ is a tautology, then in any weighted truth table, $\#(\varphi) \leq \#(\psi)$.

Since $\varphi \to \psi \approx \neg\varphi \vee \psi$, we have:

$$\begin{aligned}
\#(\top) &= \#(\varphi \to \psi) \\
&= \#(\neg\varphi \vee \psi) \\
&= \#(\neg\varphi) + \#(\psi) - \#(\neg\varphi \wedge \psi) \\
&= \#(\top) - \#(\varphi) + \#(\psi) - \#(\neg\varphi \wedge \psi)
\end{aligned}$$

So, $\#(\varphi) + \#(\neg\varphi \wedge \psi) = \#(\psi)$.

Recall that for all numbers n and m, $n \leq m$ if, and only if, there is some $x \geq 0$ such that $n + x = m$.

So, $\#(\varphi) \leq \#(\psi)$

Proportions/Probability

In a weighted truth table, the proportion or probability, that φ is true is:

$$Pr(\varphi) = \frac{\#(\varphi)}{\#(\top)}$$

	I	P
80	T	T
20	T	F
100	F	T
900	F	F

$$
\begin{aligned}
\#(P \wedge I) &= 80 & Pr(P \wedge I) &= \tfrac{80}{1100} = \tfrac{4}{55} \\
\#(P) &= 180 & Pr(P) &= \tfrac{180}{1100} = \tfrac{9}{55} \\
\#(I) &= 100 & Pr(I) &= \tfrac{100}{1100} = \tfrac{1}{11} \\
\#(\neg P) &= 1000 & Pr(\neg P) &= \tfrac{1000}{1100} = \tfrac{10}{11} \\
\#(P \vee I) &= 200 & Pr(P \vee I) &= \tfrac{200}{1100} = \tfrac{2}{11} \\
\#(P \vee \neg P) &= 1100 & Pr(P \vee \neg P) &= \tfrac{1100}{1100} = 1
\end{aligned}
$$

Truth Table

	P	Q	$\cdots$
1.	T	T	
2.	T	F	
3.	F	T	
4.	F	F	

Stochastic Truth Table

	P	Q	$\cdots$
p_1	T	T	
p_2	T	F	
p_3	F	T	
p_4	F	F	

A **Stochastic Truth Table** is a truth table where there is a number assigned to each row such that:

(1) each number is greater than or equal to 0; and

(2) and the sum of all the numbers assigned to the rows is 1

Stochastic Truth Table

	P	Q	$\cdots$
p_1	T	T	
p_2	T	F	
p_3	F	T	
p_4	F	F	

A **Stochastic Truth Table** is a truth table where each row i is assigned a number p_i such that:

(1) For each i, $p_i \geq 0$; and

(2) $\sum_i p_i = 1$

Stochastic Truth Table

	P	Q	$\cdots$
p_1	T	T	
p_2	T	F	
p_3	F	T	
p_4	F	F	

p_i is a measure of how "likely" it is that the situation described by row i obtains ("the *probability* of row i")

For example, p_2 is the measure of how likely it is that P is true and Q is false.

Stochastic Truth Table

	P	Q	$\cdots$
$\frac{2}{3}$	T	T	
$\frac{1}{6}$	T	F	
$\frac{1}{12}$	F	T	
$\frac{1}{12}$	F	F	

$\frac{2}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = 1$

The probability that P is true and Q is true is $\frac{2}{3}$.

The probability that P is true and Q is false is $\frac{1}{6}$.

The probability that P is false and Q is true is $\frac{1}{12}$.

The probability that P is false and Q is false is $\frac{1}{12}$.

Stochastic Truth Table

	P	Q	$\cdots$
$\frac{1}{4}$	T	T	
$\frac{1}{4}$	T	F	
$\frac{1}{4}$	F	T	
$\frac{1}{4}$	F	F	

$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$

The probability that P is true and Q is true is $\frac{1}{4}$.

The probability that P is true and Q is false is $\frac{1}{4}$.

The probability that P is false and Q is true is $\frac{1}{4}$.

The probability that P is false and Q is false is $\frac{1}{4}$.

Stochastic Truth Table

	P	Q	$\cdots$
$\frac{1}{4}$	T	T	
0	T	F	
0	F	T	
$\frac{3}{4}$	F	F	

$\frac{1}{4} + 0 + 0 + \frac{3}{4} = 1$

The probability that P is true and Q is true is $\frac{1}{4}$.

The probability that P is true and Q is false is 0.

The probability that P is false and Q is true is 0.

The probability that P is false and Q is false is $\frac{3}{4}$.

- *Any* assignment of nonnegative numbers that sum to 1 can be used to form a stochastic truth table.

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- Since the sum must be 1 and the numbers assigned to a row in a stochastic truth table are all greater than or equal to 0, no row is assigned a number greater than 1.

- *Any* assignment of nonnegative numbers that sum to 1 can be used to form a stochastic truth table.

- Since the sum must be 1 and the numbers assigned to a row in a stochastic truth table are all greater than or equal to 0, no row is assigned a number greater than 1.

- If a row is assigned 1, then that row is certain to obtain and all other rows must be assigned 0.

- *Any* assignment of nonnegative numbers that sum to 1 can be used to form a stochastic truth table.

- Since the sum must be 1 and the numbers assigned to a row in a stochastic truth table are all greater than or equal to 0, no row is assigned a number greater than 1.

- If a row is assigned 1, then that row is certain to obtain and all other rows must be assigned 0.

- If a row is assigned 0, then it is a logical possibility that certainly will not happen (e.g., being dealt a red spade or flipping a coin and it landing on its side).

Given a stochastic truth table, how do you determine the probability of a formula φ?

$$Pr(\varphi) = \sum\{p_i \mid \ i \text{ is a row that makes } \varphi \text{ true}\}$$

prob	A	K	Q	
$\frac{1}{8}$	T	T	T	
$\frac{1}{8}$	T	T	F	
$\frac{1}{8}$	T	F	T	
$\frac{1}{8}$	T	F	F	
$\frac{1}{8}$	F	T	T	
$\frac{1}{8}$	F	T	F	
$\frac{1}{8}$	F	F	T	
$\frac{1}{8}$	F	F	F	

$$
\begin{aligned}
Pr(A) &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
&= \frac{4}{8} = \frac{1}{2}
\end{aligned}
$$

prob	A	K	Q	$A \wedge K$
$\frac{1}{8}$	T	T	T	T
$\frac{1}{8}$	T	T	F	T
$\frac{1}{8}$	T	F	T	F
$\frac{1}{8}$	T	F	F	F
$\frac{1}{8}$	F	T	T	F
$\frac{1}{8}$	F	T	F	F
$\frac{1}{8}$	F	F	T	F
$\frac{1}{8}$	F	F	F	F

$$\begin{aligned} Pr(A \wedge K) &= \frac{1}{8} + \frac{1}{8} \\ &= \frac{2}{8} = \frac{1}{4} \end{aligned}$$

prob	A	K	Q	$K \to Q$
$\frac{1}{8}$	T	T	T	T
$\frac{1}{8}$	T	T	F	F
$\frac{1}{8}$	T	F	T	T
$\frac{1}{8}$	T	F	F	T
$\frac{1}{8}$	F	T	T	T
$\frac{1}{8}$	F	T	F	F
$\frac{1}{8}$	F	F	T	T
$\frac{1}{8}$	F	F	F	T

$$Pr(K \to Q) = 6 * \frac{1}{8}$$
$$= \frac{6}{8} = \frac{3}{4}$$

prob	A	K	Q	$K \to Q$
$\frac{1}{8}$	T	T	T	T
$\frac{1}{8}$	T	T	F	F
$\frac{1}{8}$	T	F	T	T
$\frac{1}{8}$	T	F	F	T
$\frac{1}{8}$	F	T	T	T
$\frac{1}{8}$	F	T	F	F
$\frac{1}{8}$	F	F	T	T
$\frac{1}{8}$	F	F	F	T

$$Pr(K \to Q) = 6 * \frac{1}{8} = \frac{6}{8} = \frac{3}{4}$$

prob	A	K	Q	$K \rightarrow Q$
$\frac{1}{8}$	T	T	T	T
$\frac{1}{8}$	T	T	F	F
$\frac{1}{8}$	T	F	T	T
$\frac{1}{8}$	T	F	F	T
$\frac{1}{8}$	F	T	T	T
$\frac{1}{8}$	F	T	F	F
$\frac{1}{8}$	F	F	T	T
$\frac{1}{8}$	F	F	F	T

$$\begin{aligned} Pr(K \rightarrow Q) &= 6 * \tfrac{1}{8} \\ &= \tfrac{6}{8} = \tfrac{3}{4} \end{aligned}$$

prob	A	K	Q	
$\frac{1}{8}$	T	T	T	
$\frac{1}{8}$	T	T	F	
$\frac{1}{8}$	T	F	T	
$\frac{1}{8}$	T	F	F	
$\frac{1}{8}$	F	T	T	
$\frac{1}{8}$	F	T	F	
$\frac{1}{8}$	F	F	T	
$\frac{1}{8}$	F	F	F	

$$Pr(K \wedge \neg K) = 0$$

$$Pr((K \wedge Q) \rightarrow Q) = 1$$

$$Pr(\neg K \wedge \neg A) = \frac{2}{8} = \frac{1}{4}$$

$$Pr(\neg K \vee Q) = \frac{6}{8} = \frac{3}{4}$$

prob	A	K	Q	
$\frac{1}{8}$	T	T	T	
$\frac{1}{8}$	T	T	F	
$\frac{1}{8}$	T	F	T	
$\frac{1}{8}$	T	F	F	
$\frac{1}{8}$	F	T	T	
$\frac{1}{8}$	F	T	F	
$\frac{1}{8}$	F	F	T	
$\frac{1}{8}$	F	F	F	

$$
\begin{aligned}
Pr(K \wedge \neg K) &= 0 \\
Pr((K \wedge Q) \rightarrow Q) &= 1 \\
Pr(\neg K \wedge \neg A) &= \frac{2}{8} = \frac{1}{4} \\
Pr(\neg K \vee Q) &= \frac{6}{8} = \frac{3}{4}
\end{aligned}
$$

prob	A	K	Q	
$\frac{1}{8}$	T	T	T	
$\frac{1}{8}$	T	T	F	
$\frac{1}{8}$	T	F	T	
$\frac{1}{8}$	T	F	F	
$\frac{1}{8}$	F	T	T	
$\frac{1}{8}$	F	T	F	
$\frac{1}{8}$	F	F	T	
$\frac{1}{8}$	F	F	F	

$$
\begin{aligned}
Pr(K \wedge \neg K) &= 0 \\
Pr((K \wedge Q) \rightarrow Q) &= 1 \\
Pr(\neg K \wedge \neg A) &= \tfrac{2}{8} = \tfrac{1}{4} \\
Pr(\neg K \vee Q) &= \tfrac{6}{8} = \tfrac{3}{4}
\end{aligned}
$$