Reasoning for Humans: Clear Thinking in an Uncertain World

PHIL 171

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Summary

Formulas

1. Every atomic formula is a formula of sentential logic.
2. If X is a formula of sentential logic, then so is $\neg X$.
3. If X and Y are formulas of sentential logic, then so are each of the following:
   a. $(X \wedge Y)$
   b. $(X \vee Y)$
   c. $(X \rightarrow Y)$
4. An expression of sentential logic is a formula only if it can be constructed by one or more applications of the first three rules.

Main Connective

1. If the first symbol in the formula is "$\neg$", then the first occurrence of "$\neg$" is the main connective.
2. Otherwise, the main connective is the first occurrence of the binary connective that is surrounded by the fewest number of parentheses.

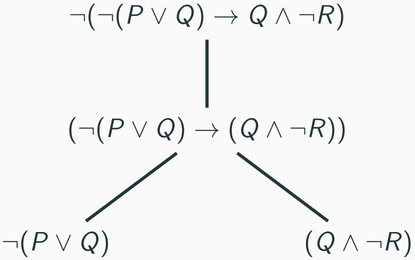
Syntax Tree

1. Write down the formula (adding parentheses if necessary).
2. Identify the main connective.
3. If it is a negation, write the negated formula below the current formula.
4. If it is a disjunction/conjunction/conditional, write the left disjunct/left conjunct/antecedent down left of the current formula and the right disjunct/right conjunct/consequent below right of the current formula.
5. Repeat steps 2-4 for for every formula that is not an atomic formula.
6. Draw lines connecting formula to the ones immediately below them.

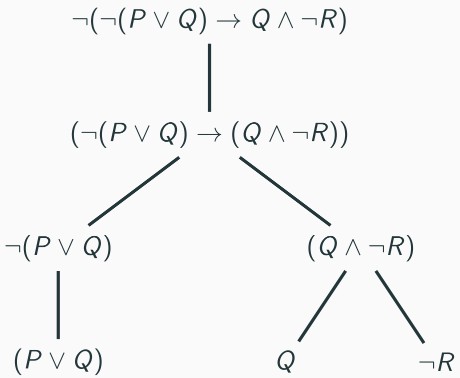
$$\neg(\neg(P \vee Q) \rightarrow Q \wedge \neg R)$$

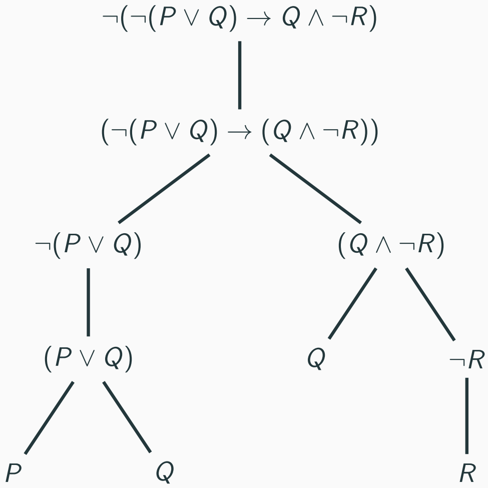
$$\neg(\neg(P \vee Q) \rightarrow Q \wedge \neg R)$$

$$(\neg(P \vee Q) \rightarrow (Q \wedge \neg R))$$



¬(¬(P ∨ Q) → Q ∧ ¬R)
(¬(P ∨ Q) → (Q ∧ ¬R))
¬(P ∨ Q)
(Q ∧ ¬R)


¬(¬(P ∨ Q) → Q ∧ ¬R)
(¬(P ∨ Q) → (Q ∧ ¬R))
¬(P ∨ Q)
(Q ∧ ¬R)
(P ∨ Q)
Q
¬R



$\neg(\neg(P \vee Q) \rightarrow Q \wedge \neg R)$
$(\neg(P \vee Q) \rightarrow (Q \wedge \neg R))$
$\neg(P \vee Q)$
$(Q \wedge \neg R)$
$(P \vee Q)$
Q
$\neg R$
P
Q
R

Truth Tables

X	Y	$(X \wedge Y)$
T	T	T
T	F	F
F	T	F
F	F	F

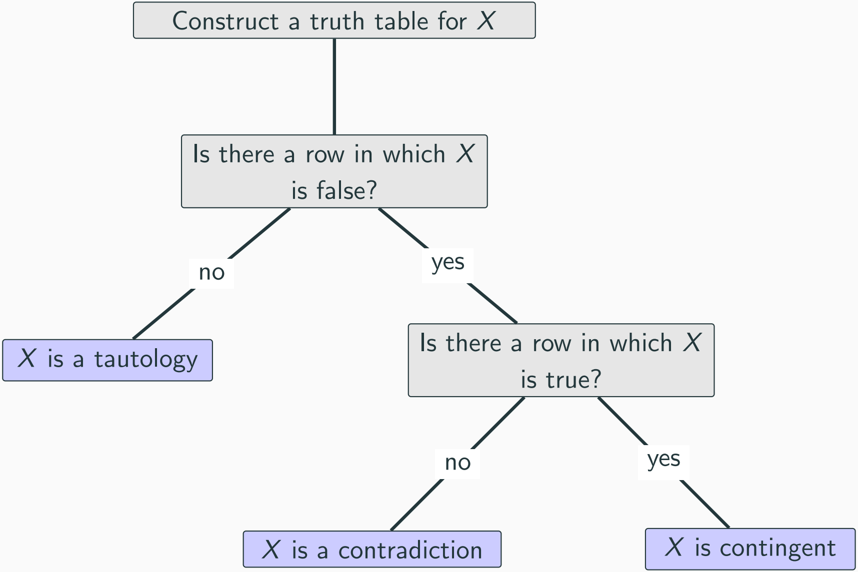
X	Y	$(X \vee Y)$
T	T	T
T	F	T
F	T	T
F	F	F

X	Y	$(X \leftrightarrow Y)$
T	T	T
T	F	F
F	T	F
F	F	T

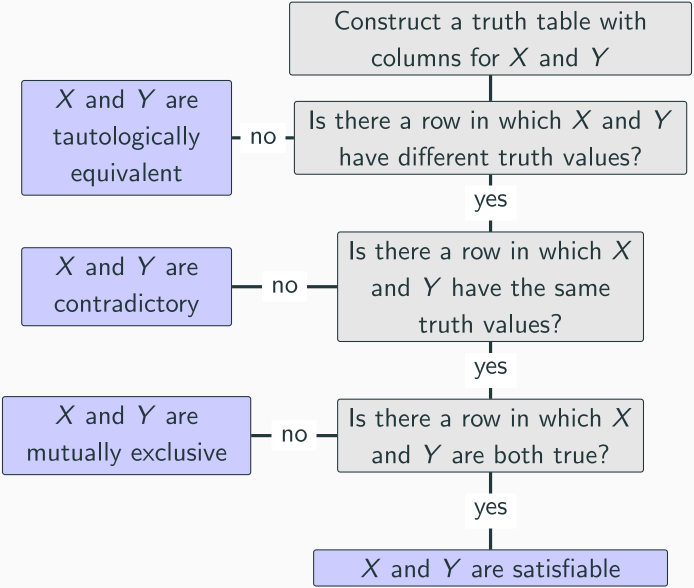
X	Y	$(X \rightarrow Y)$
T	T	T
T	F	F
F	T	T
F	F	T

X	$\neg X$
T	F
F	T

Classifying a Formula



Classifying Sets of Formulas



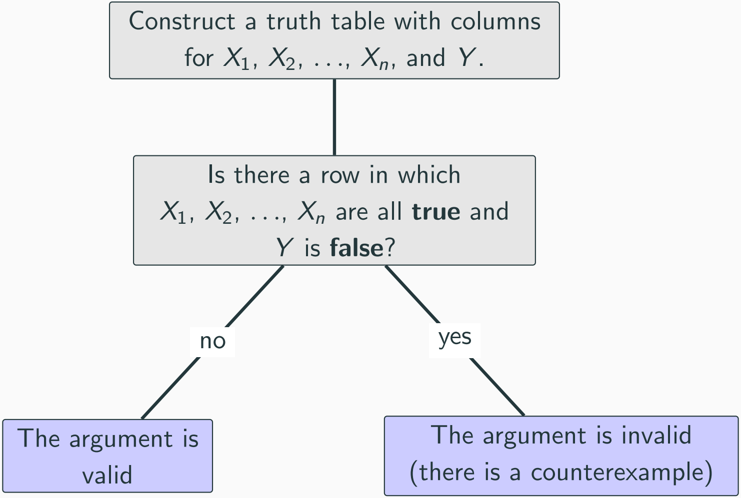
Valid Arguments

$X_1, \ldots, X_n \Rightarrow Y$:
An argument with premises $X_1, \ldots, X_n$ and conclusion Y.

$X_1, \ldots, X_n \models Y$:
A valid argument with premises $X_1, \ldots, X_n$ and conclusion Y.

$X_1, \ldots, X_n \not\models Y$:
An invalid argument with premises $X_1, \ldots, X_n$ and conclusion Y.

Valid Arguments



$$A \to C, B \to C, A \lor B \Rightarrow C$$

A	B	C	$A \to C$	$B \to C$	$A \lor B$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

$$A \to C, B \to C, A \vee B \Rightarrow C$$

A	B	C	$A \to C$	$B \to C$	$A \vee B$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	F
F	F	F	T	T	F

$$A \to C, B \to C, A \vee B \Rightarrow C$$

A	B	C	$A \to C$	$B \to C$	$A \vee B$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	F
F	F	F	T	T	F

$$A \rightarrow C, B \rightarrow C, A \vee B \Rightarrow C$$

A	B	C	$A \rightarrow C$	$B \rightarrow C$	$A \vee B$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	F
F	F	F	T	T	F

$$A \to C, B \to C, A \lor B \models C$$

This argument is valid because there is no truth-value assignment that makes the premises true ($A \to C$, $B \to C$ and $A \lor B$) and the conclusion (C) false.

$$A \to C, B \to C, A \vee B \models C$$

A	B	C	$A \to C$	$B \to C$	$A \vee B$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	F
F	F	F	T	T	F

$$A \to C, B \to C, A \vee B \models C$$

A	B	C	$A \to C$	$B \to C$	$A \vee B$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	F
F	F	F	T	T	F

$$A \to C, B \to C, A \vee B \models C$$

This argument is valid because in every row in which the conclusion (C) is false, at least one of the premises ($A \to C$, $B \to C$ or $A \vee B$) is false.

Is the following argument valid or invalid? You must show your answer.

$$(A \vee B), (B \to C) \Rightarrow (B \wedge C)$$

$$(A \vee B), (B \to C) \Rightarrow (B \wedge C)$$

A	B	C	$(A \vee B)$	$(B \to C)$	$(B \wedge C)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	T	T	F
T	F	F	T	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	F	T	F

$$(A \vee B), (B \to C) \not\models (B \wedge C)$$

A	B	C	$(A \vee B)$	$(B \to C)$	$(B \wedge C)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	T	T	F
T	F	F	T	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	F	T	F

The argument is invalid, because there is a counterexample.

Valid Inference Rules

Name	Valid inference rule
Modus Ponens	$X, X \rightarrow Y \models Y$
Modus Tollens	$X \rightarrow Y, \neg Y \models \neg X$
Disjunctive Syllogism	$X \vee Y, \neg X \models Y$
Transitivity	$X \rightarrow Y, Y \rightarrow Z \models X \rightarrow Z$

Observation. If X and Y are tautologically equivalent, denoted $X \approx Y$, then for all formulas Z,

$X \models Z$ if and only if $Y \models Z$.

$Z \models X$ if and only if $Z \models Y$.

Boolean Equivalences

Name	Equivalence
Double Negation	$\neg\neg X \approx X$
DeMorgan's Rules	$\neg(X \vee Y) \approx (\neg X \wedge \neg Y)$ $\neg(X \wedge Y) \approx (\neg X \vee \neg Y)$
Distribution	$(X \wedge (Y \vee Z)) \approx ((X \wedge Y) \vee (X \wedge Z))$ $(X \vee (Y \wedge Z)) \approx ((X \vee Y) \wedge (X \vee Z))$
Absorption	$(X \vee (X \wedge Y) \approx X$ $(X \wedge (X \vee Y) \approx X$
Commutativity	$(X \wedge Y) \approx (Y \wedge X)$ $(X \vee Y) \approx (Y \vee X)$
Conditional Definition	$(X \rightarrow Y) \approx (\neg X \vee Y)$

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$(P \wedge Q) \vee (P \wedge \neg Q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	F	F
F	F	F	F	F

$$P \approx (P \wedge Q) \vee (P \wedge \neg Q)$$

$$P \approx (P \wedge Q) \vee (P \wedge \neg Q).$$

$$P \approx (P \wedge Q) \vee (P \wedge \neg Q).$$

Tautology Rule: If Y is a tautology, then $X \approx (X \wedge Y)$

$$P \approx (P \wedge Q) \vee (P \wedge \neg Q).$$

Tautology Rule: If Y is a tautology, then $X \approx (X \wedge Y)$

1. $P \approx P \wedge (Q \vee \neg Q)$

   Since $Q \vee \neg Q$ is a tautology, this is an instance of the Tautology Rule.

2. $P \wedge (Q \vee \neg Q) \approx (P \wedge Q) \vee (P \wedge \neg Q)$

   This is an instance of one of the Distribution axioms.

$$P \approx (P \wedge Q) \vee (P \wedge \neg Q).$$

Tautology Rule: If Y is a tautology, then $X \approx (X \wedge Y)$

1. $P \approx P \wedge (Q \vee \neg Q)$

   Since $Q \vee \neg Q$ is a tautology, this is an instance of the Tautology Rule.

2. $P \wedge (Q \vee \neg Q) \approx (P \wedge Q) \vee (P \wedge \neg Q)$

   This is an instance of one of the Distribution axioms.

Transitivity Rule: If $X \approx Y$ and $Y \approx Z$, then $X \approx Z$

3. $P \approx (P \wedge Q) \vee (P \wedge \neg Q)$

   This follows from the Transitivity Rule since we established in step 1 that $P \approx P \wedge (Q \vee \neg Q)$ and in step 2 that $P \wedge (Q \vee \neg Q) \approx (P \wedge Q) \vee (P \wedge \neg Q)$.

We can adapt the previous explanation to show the following: for any formulas X and Y,

$$X \approx (X \wedge Y) \vee (X \wedge \neg Y).$$

Name	Equivalence
Reflexivity	$X \approx X$
Symmetry	If $X \approx Y$, then $Y \approx X$
Transitivity	If $X \approx Y$ and $Y \approx Z$, then $X \approx Z$
Tautology	If Y is a tautology, then $X \approx (X \wedge Y)$
Contradiction	If Y is a contradiction, then $X \approx (X \vee Y)$

Venn Diagrams

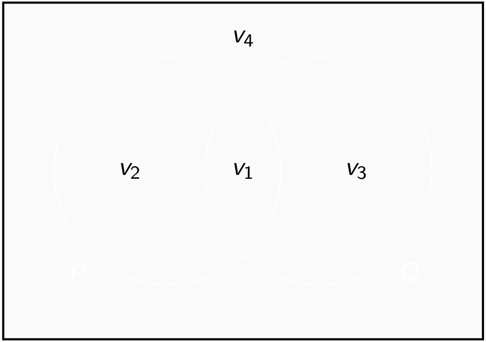
P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Venn Diagrams

	P	Q	$P \vee Q$
v_1	T	T	T
v_2	T	F	T
v_3	F	T	T
v_4	F	F	F

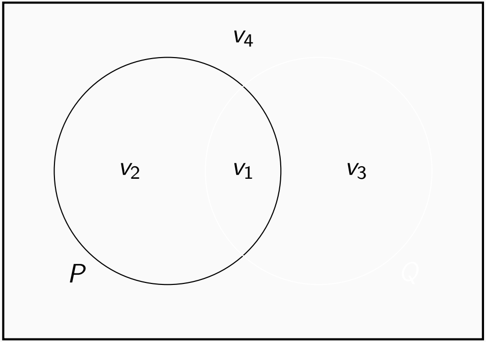
Venn Diagrams

	P	Q	$P \vee Q$
v_1	T	T	T
v_2	T	F	T
v_3	F	T	T
v_4	F	F	F



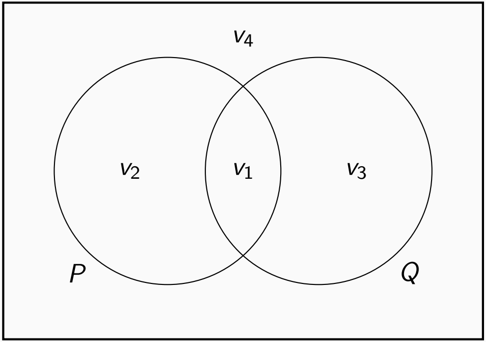
Venn Diagrams

	P	Q	$P \vee Q$
v_1	T	T	T
v_2	T	F	T
v_3	F	T	T
v_4	F	F	F



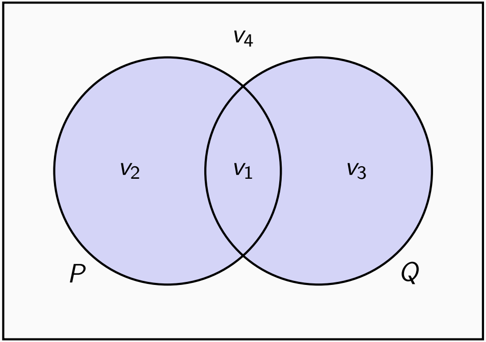
Venn Diagrams

	P	Q	$P \vee Q$
v_1	T	T	T
v_2	T	F	T
v_3	F	T	T
v_4	F	F	F



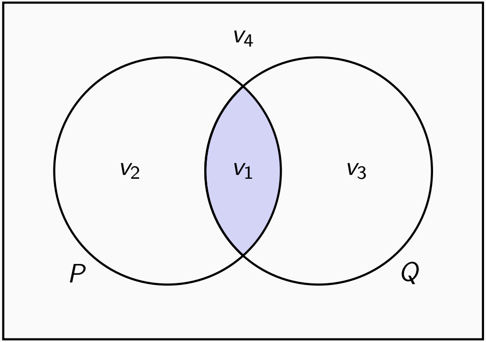
Venn Diagrams

	P	Q	$P \vee Q$
v_1	T	T	T
v_2	T	F	T
v_3	F	T	T
v_4	F	F	F



Venn Diagrams

	P	Q	$P \wedge Q$
v_1	T	T	T
v_2	T	F	F
v_3	F	T	F
v_4	F	F	F



Venn Diagrams

	P	Q	$\neg P$
v_1	T	T	F
v_2	T	F	F
v_3	F	T	T
v_4	F	F	T