Reasoning for Humans: Clear Thinking in an Uncertain World

PHIL 171

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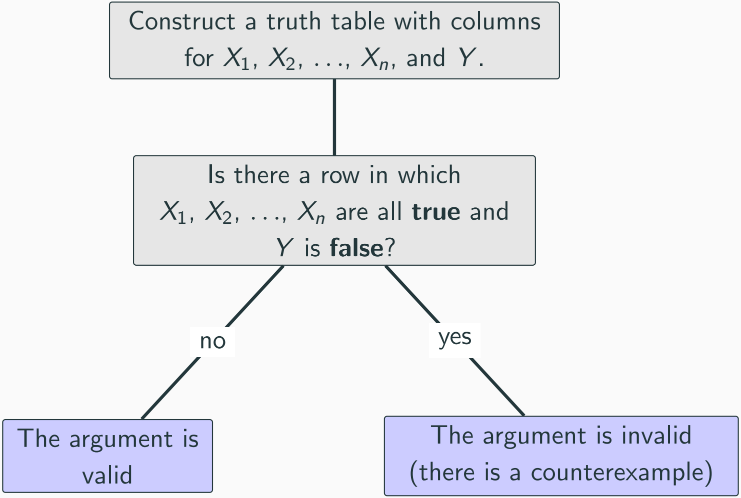
Recap: Truth Tables

X	Y	$(X \wedge Y)$
T	T	T
T	F	F
F	T	F
F	F	F

X	Y	$(X \vee Y)$
T	T	T
T	F	T
F	T	T
F	F	F

X	Y	$(X \rightarrow Y)$
T	T	T
T	F	F
F	T	T
F	F	T

X	$\neg X$
T	F
F	T



Construct a truth table with columns for X_1, X_2, . . ., X_n, and Y.
Is there a row in which X_1, X_2, . . ., X_n are all **true** and Y is **false**?
no
yes
The argument is valid
The argument is invalid (there is a counterexample)

Notation

$X_1, \ldots, X_n \Rightarrow Y$:
An argument with premises $X_1, \ldots, X_n$ and conclusion Y.

$X_1, \ldots, X_n \models Y$:
A valid argument with premises $X_1, \ldots, X_n$ and conclusion Y.

$X_1, \ldots, X_n \not\models Y$:
An invalid argument with premises $X_1, \ldots, X_n$ and conclusion Y.

Name	Valid inference rule
Modus Ponens	$X, X \rightarrow Y \models Y$
Modus Tollens	$X \rightarrow Y, \neg Y \models \neg X$
Disjunctive Syllogism	$X \vee Y, \neg X \models Y$
Transitivity	$X \rightarrow Y, Y \rightarrow Z \models X \rightarrow Z$

Is $(Q \wedge P), (P \wedge Q) \to S \Rightarrow S$ a valid argument?

Yes: $(Q \wedge P), (P \wedge Q) \to S \models S$

Is $(Q \wedge P), (P \wedge Q) \to S \models S$ an instance of Modus Ponens?

No: $(Q \wedge P), (P \wedge Q) \to S \models S$ is **not** an instance of Modus Ponens $X, X \to Y \models Y$.

The problem: $Q \wedge P$ and $P \wedge Q$ are **not** the same formula.

However, $Q \wedge P$ and $P \wedge Q$ are tautologically equivalent.

For any formulas X and Y, we write $X \approx Y$ when X and Y are tautologically equivalent.

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$$(P \wedge Q) \approx (Q \wedge P)$$

P	Q	$P \wedge Q$	$Q \wedge P$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

Since $(P \wedge Q), (P \wedge Q) \rightarrow S \models S$ is an instance of Modus Ponens and $(Q \wedge P) \approx (P \wedge Q)$, we can conclude that

$$(Q \wedge P), (P \wedge Q) \rightarrow S \models S$$

Since $\neg\neg Q, P \to \neg Q \models \neg P$ is an instance of Modus Tollens and $Q \approx \neg\neg Q$, we can conclude that

$$Q, P \to \neg Q \models \neg P$$

Since $P, P \to Q \models Q$ is an instance of Modus Ponens and $\neg P \lor Q \approx P \to Q$, we can conclude that

$$P, \neg P \lor Q \models Q$$

Observation. If X and Y are tautologically equivalent, denoted $X \approx Y$, then for all formulas Z,

$X \models Z$ if and only if $Y \models Z$.

$Z \models X$ if and only if $Z \models Y$.

Boolean Equivalences

Name	Equivalence
Double Negation	$\neg\neg X \approx X$
DeMorgan's Rules	$\neg(X \vee Y) \approx (\neg X \wedge \neg Y)$ $\neg(X \wedge Y) \approx (\neg X \vee \neg Y)$
Distribution	$(X \wedge (Y \vee Z)) \approx ((X \wedge Y) \vee (X \wedge Z))$ $(X \vee (Y \wedge Z)) \approx ((X \vee Y) \wedge (X \vee Z))$
Absorption	$(X \vee (X \wedge Y) \approx X$ $(X \wedge (X \vee Y) \approx X$
Commutativity	$(X \wedge Y) \approx (Y \wedge X)$ $(X \vee Y) \approx (Y \vee X)$
Conditional Definition	$(X \rightarrow Y) \approx (\neg X \vee Y)$

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$(P \wedge Q) \vee (P \wedge \neg Q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	F	F
F	F	F	F	F

$$P \approx (P \wedge Q) \vee (P \wedge \neg Q)$$

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Tautology Rule: If Y is a tautology, then $X \approx (X \wedge Y)$

$$P \approx (P \wedge Q) \vee (P \wedge \neg Q).$$

Tautology Rule: If Y is a tautology, then $X \approx (X \wedge Y)$

1. $P \approx P \wedge (Q \vee \neg Q)$

   Since $Q \vee \neg Q$ is a tautology, this is an instance of the Tautology Rule.

2. $P \wedge (Q \vee \neg Q) \approx (P \wedge Q) \vee (P \wedge \neg Q)$

   This is an instance of one of the Distribution axioms.

$$P \approx (P \wedge Q) \vee (P \wedge \neg Q).$$

Tautology Rule: If Y is a tautology, then $X \approx (X \wedge Y)$

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2. $P \wedge (Q \vee \neg Q) \approx (P \wedge Q) \vee (P \wedge \neg Q)$

   This is an instance of one of the Distribution axioms.

Transitivity Rule: If $X \approx Y$ and $Y \approx Z$, then $X \approx Z$

3. $P \approx (P \wedge Q) \vee (P \wedge \neg Q)$

   This follows from the Transitivity Rule since we established in step 1 that $P \approx P \wedge (Q \vee \neg Q)$ and in step 2 that $P \wedge (Q \vee \neg Q) \approx (P \wedge Q) \vee (P \wedge \neg Q)$.

We can adapt the previous explanation to show the following: for any formulas X and Y,

$$X \approx (X \wedge Y) \vee (X \wedge \neg Y).$$

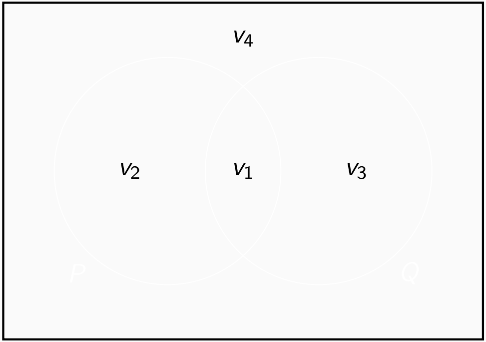
Name	Equivalence
Reflexivity	$X \approx X$
Symmetry	If $X \approx Y$, then $Y \approx X$
Transitivity	If $X \approx Y$ and $Y \approx Z$, then $X \approx Z$
Tautology	If Y is a tautology, then $X \approx (X \wedge Y)$
Contradiction	If Y is a contradiction, then $X \approx (X \vee Y)$

Visualizing propositions

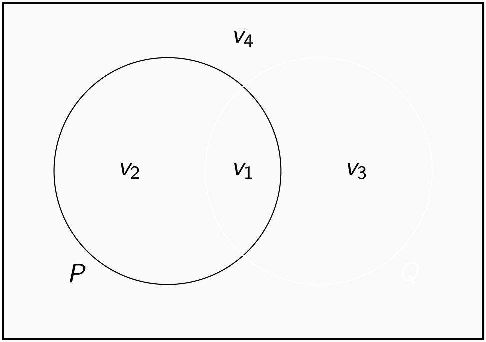
P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

	P	Q	$P \vee Q$
v_1	T	T	T
v_2	T	F	T
v_3	F	T	T
v_4	F	F	F

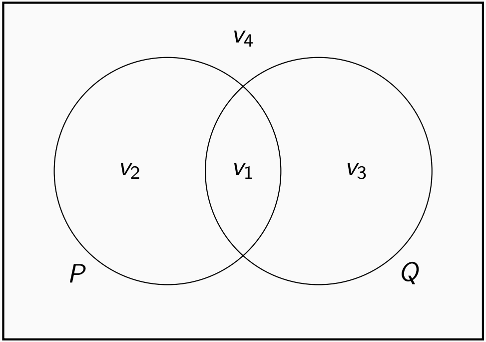
	P	Q	$P \vee Q$
v_1	T	T	T
v_2	T	F	T
v_3	F	T	T
v_4	F	F	F



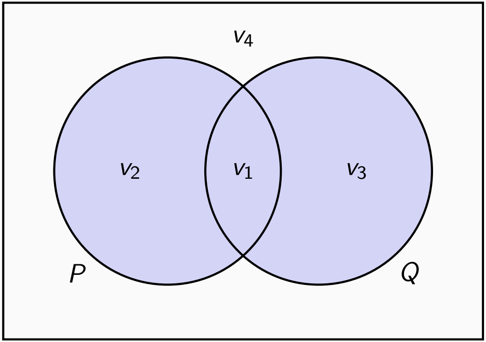
	P	Q	$P \vee Q$
v_1	T	T	T
v_2	T	F	T
v_3	F	T	T
v_4	F	F	F



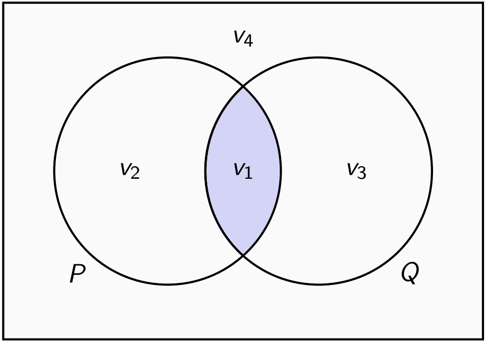
	P	Q	$P \vee Q$
v_1	T	T	T
v_2	T	F	T
v_3	F	T	T
v_4	F	F	F



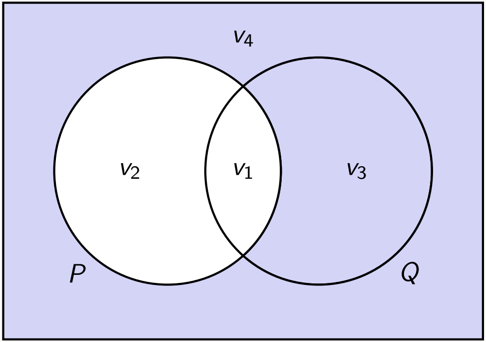
	P	Q	$P \vee Q$
v_1	T	T	T
v_2	T	F	T
v_3	F	T	T
v_4	F	F	F

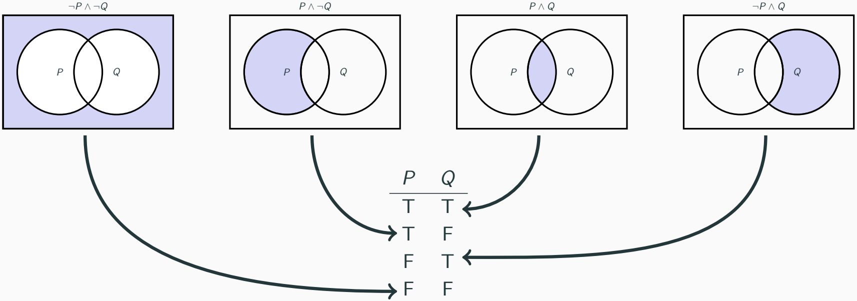


	P	Q	$P \wedge Q$
v_1	T	T	T
v_2	T	F	F
v_3	F	T	F
v_4	F	F	F



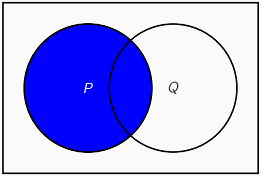
	P	Q	$\neg P$
v_1	T	T	F
v_2	T	F	F
v_3	F	T	T
v_4	F	F	T



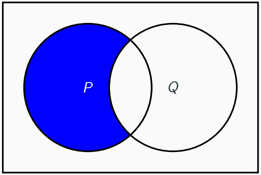

¬P ∧ ¬Q
P ∧ ¬Q
P ∧ Q
¬P ∧ Q
P
Q
P Q
T T
T F
F T
F F

$$P \approx (P \wedge Q) \vee (P \wedge \neg Q)$$

P



$P \wedge \neg Q$



$P \wedge Q$