Reasoning for Humans: Clear Thinking in an Uncertain World

PHIL 171

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Recap: Truth Tables

X	Y	$(X \wedge Y)$
T	T	T
T	F	F
F	T	F
F	F	F

X	Y	$(X \vee Y)$
T	T	T
T	F	T
F	T	T
F	F	F

X	Y	$(X \to Y)$
T	T	T
T	F	F
F	T	T
F	F	T

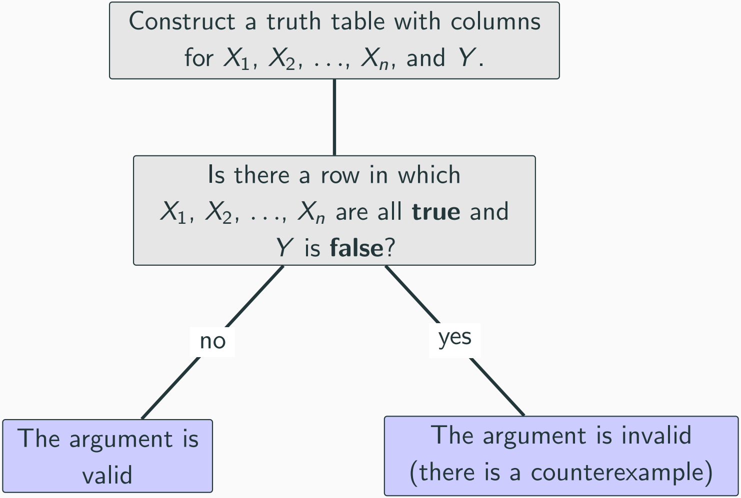
X	$\neg X$
T	F
F	T

Valid Argument: An argument is valid provided that there is no truth value assignment that makes all the premises true and the conclusion false. (So, any truth-value assignment that makes all the premises true also makes the conclusion true).

Invalid Argument: An argument is invalid just in case it is not valid, i.e., if there is some truth-value assignment that makes the premises true and the conclusion false.

Counterexample: A truth-value assignment that makes the premises of an argument true and its conclusion false is called a counterexample to the argument.

So, an argument if valid if there are no counterexamples.


Construct a truth table with columns for X_1, X_2, . . ., X_n, and Y.
Is there a row in which X_1, X_2, . . ., X_n are all **true** and Y is **false**?
no
yes
The argument is valid
The argument is invalid (there is a counterexample)

Notation

$X_1, \ldots, X_n \Rightarrow Y$:
An argument with premises $X_1, \ldots, X_n$ and conclusion Y.

$X_1, \ldots, X_n \models Y$:
A valid argument with premises $X_1, \ldots, X_n$ and conclusion Y.

$X_1, \ldots, X_n \not\models Y$:
An invalid argument with premises $X_1, \ldots, X_n$ and conclusion Y.

Weird Cases: Which arguments are valid?

$$B \Rightarrow (A \vee \neg A)$$

$$(A \vee \neg A) \Rightarrow B$$

$$(A \wedge \neg A) \Rightarrow B$$

$$A \Rightarrow (B \wedge \neg B)$$

$$(A \wedge \neg A) \Rightarrow (B \wedge \neg B)$$

Weird Cases

- Suppose that an argument contains a tautology as a premise. Is the argument valid?

Weird Cases

- Suppose that an argument contains a tautology as a premise. Is the argument valid? Maybe.

Weird Cases

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- Suppose that the argument contains a contradiction as a premise. Is the argument valid?

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Weird Cases

- Suppose that an argument contains a tautology as a premise. Is the argument valid? Maybe.

- Suppose that the argument contains a contradiction as a premise. Is the argument valid? Yes.

- Suppose that the argument contains a tautology as a conclusion. Is the argument valid? Yes.

Weird Cases

- Suppose that an argument contains a tautology as a premise. Is the argument valid? Maybe.
- Suppose that the argument contains a contradiction as a premise. Is the argument valid? Yes.
- Suppose that the argument contains a tautology as a conclusion. Is the argument valid? Yes.
- Suppose that the argument contains a contradiction as a conclusion. Is the argument valid?

Weird Cases

- Suppose that an argument contains a tautology as a premise. Is the argument valid? Maybe.
- Suppose that the argument contains a contradiction as a premise. Is the argument valid? Yes.
- Suppose that the argument contains a tautology as a conclusion. Is the argument valid? Yes.
- Suppose that the argument contains a contradiction as a conclusion. Is the argument valid? Maybe.

Weird Cases

$$B \models (A \vee \neg A)$$

$$(A \vee \neg A) \not\models B$$

$$(A \wedge \neg A) \models B$$

$$A \not\models (B \wedge \neg B)$$

$$(A \wedge \neg A) \models (B \wedge \neg B)$$

$X, X \to Y \models Y$

$P, P \to Q \Rightarrow Q$

$A, A \to B \Rightarrow B$

$Q, Q \to P \Rightarrow P$

$\neg A, \neg A \to B \Rightarrow B$

$(Q \vee R), (Q \vee R) \to P \Rightarrow P$

$(P \to Q), (P \to Q) \to (R \vee (Q \to S)) \Rightarrow (R \vee (Q \to S))$

$X, X \to Y \models Y$

$P, P \to Q \models Q$

$A, A \to B \models B$

$Q, Q \to P \models P$

$\neg A, \neg A \to B \models B$

$(Q \vee R), (Q \vee R) \to P \models P$

$(P \to Q), (P \to Q) \to (R \vee (Q \to S)) \models (R \vee (Q \to S))$

$$X, X \to Y \models Y$$

$P, P \to Q \models Q$

$A, A \to B \models B$

$Q, Q \to P \models P$

$\neg A, \neg A \to B \models B$

$(Q \vee R), (Q \vee R) \to P \models P$

$(P \to Q), (P \to Q) \to (R \vee (Q \to S)) \models (R \vee (Q \to S))$

$$X, X \to Y \models Y$$

Every way of replacing X with a formula and Y with a formula results in a valid argument.

Name	Valid inference rule
Modus Ponens	$X, X \rightarrow Y \models Y$
Modus Tollens	$X \rightarrow Y, \neg Y \models \neg X$
Disjunctive Syllogism	$X \vee Y, \neg X \models Y$
Transitivity	$X \rightarrow Y, Y \rightarrow Z \models X \rightarrow Z$

$$Q, P \to Q \Rightarrow P$$

$$B, A \to B \Rightarrow A$$

$$P, Q \to P \Rightarrow Q$$

$$B, \neg A \to B \Rightarrow \neg A$$

$$P, (Q \vee R) \to P \Rightarrow (Q \vee R)$$

$$(R \vee (Q \to S)), (P \to Q) \to (R \vee (Q \to S)) \Rightarrow (P \to Q)$$

$$Q, P \to Q \not\models P$$

$$B, A \to B \not\models A$$

$$P, Q \to P \not\models Q$$

$$B, \neg A \to B \not\models \neg A$$

$$P, (Q \vee R) \to P \not\models (Q \vee R)$$

$$(R \vee (Q \to S)), (P \to Q) \to (R \vee (Q \to S)) \not\models (P \to Q)$$

$Q, P \to Q \not\models P$

$B, A \to B \not\models A$

$P, Q \to P \not\models Q$

$B, \neg A \to B \not\models \neg A$

$P, (Q \vee R) \to P \not\models (Q \vee R)$

$(R \vee (Q \to S)), (P \to Q) \to (R \vee (Q \to S)) \not\models (P \to Q)$

$$X, X \to Y \not\models Y$$

Some way of replacing X with a formula and Y with a formula results in an invalid argument.

Name	Invalid inference rule
Denying the Antecedent	$\neg X, X \rightarrow Y \not\models \neg Y$
Affirming the Consequent	$Y, X \rightarrow Y \not\models X$
Affirming a Disjunct	$X \vee Y, X \not\models \neg Y$

left-$\wedge$ weakening: $X \wedge Y \models X$

$P \wedge Q \models P$: The **right-$\wedge$ elimination** rule transforms $P \wedge Q$ into P. In this case, we say:

"P is inferred from $P \wedge Q$ using the **right-$\wedge$ elimination** inference rule"

1. $(P \wedge Q)$ is inferred from $(P \wedge Q) \wedge R$ using right-$\wedge$ elimination
2. P is inferred from $P \wedge P$ using right-$\wedge$ elimination
3. $(P \rightarrow \neg Q)$ is inferred from $(P \rightarrow \neg Q) \wedge (S \vee \neg R)$ using right-$\wedge$ elimination

left-$\wedge$ **weakening**: $X \wedge Y \models X$

Note that the following are **not** instances of right-$\wedge$ weakening:

1. Infer P from $(P \wedge Q) \wedge R$, denoted $(P \wedge Q) \wedge R \Rightarrow P$:
   P requires **two application** of right-$\wedge$ elimination.

2. Infer Q from $P \wedge Q$, denoted $P \wedge Q \Rightarrow Q$:
   This is an instance of a related rule "left-$\wedge$ elimination: $X \wedge Y \rightarrow Y$.

$$(Q \wedge P), (P \wedge Q) \rightarrow S \Rightarrow S$$

$$(Q \wedge P) \approx (P \wedge Q)$$

$$Q, P \rightarrow \neg Q \Rightarrow \neg P$$

$$Q \approx \neg\neg Q$$

$$P, \neg P \vee Q \Rightarrow Q$$

$$\neg P \vee Q \approx P \rightarrow Q$$

P	Q	$P \wedge Q$	$Q \wedge P$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

P	Q	$\neg\neg Q$
T	T	T
T	F	F
F	T	T
F	F	F

P	Q	$\neg P \vee Q$	$P \rightarrow Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

$(Q \wedge P), (P \wedge Q) \rightarrow S \models S$

Not an application of Modus Ponens

Replace $Q \wedge P$ with a tautologically equivalent formula $P \wedge Q$, *then* apply Modus Ponens

$Q, P \rightarrow \neg Q \models \neg P$

Not an application of Modus Tollens

Replace Q with a tautologically equivalent formula $\neg\neg Q$, *then* apply Modus Tolens

$P, \neg P \vee Q \models Q$

Not an application of Modus Ponens

Replace $\neg P \vee Q$ with a tautologically equivalent formula $P \rightarrow Q$, *then* apply Modus Ponens

For any formulas X and Y, we write $X \approx Y$ when X and Y are tautologically equivalent.

$$(Q \wedge P), (P \wedge Q) \to S \models S$$

$$(Q \wedge P) \approx (P \wedge Q)$$

$$(P \wedge Q), (P \wedge Q) \to S \models S$$

$$Q, P \to \neg Q \models \neg P$$

$$Q \approx \neg\neg Q$$

$$\neg\neg Q, P \to \neg Q \models \neg P$$

$$P, \neg P \vee Q \models Q$$

$$\neg P \vee Q \approx P \to Q$$

$$P, P \to Q \models Q$$