Reasoning for Humans: Clear Thinking in an Uncertain World

PHIL 171

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Propositional Logic

- ✓ Boolean Connectives
- Formulas
- Translation
- Truth of Formulas
- Truth Tables
- Classifying Formulas
- Valid Arguments

Boolean Connectives

English expression	Logical connective
not, it is not the case that, it is false that	$\neg$
and, yet, but, however, both, also, although, nevertheless, still, also, although, moreover, additionally, furthermore	$\wedge$
or, unless, either ... or ...	$\vee$
if ... then ..., only if, given that, in case, provided that, on condition that, sufficient condition, necessary condition, unless	$\rightarrow$

For all statements X and Y, we write:

- "$X \wedge Y$" instead of "X and Y".
- "$X \vee Y$" instead of "X or Y".
- "$\neg X$" instead of " not X".
- "$X \rightarrow Y$" instead of "if X then Y".

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- "$X \wedge Y$" instead of "X and Y".
- "$X \vee Y$" instead of "X or Y".
- "$\neg X$" instead of " not X".
- "$X \rightarrow Y$" instead of "if X then Y".

So, the abstract argument:

$$X \text{ or } Y, \text{ not } X \Rightarrow Y$$

will be written as:

$$X \vee Y, \neg X \Rightarrow Y.$$

Formulas are constructed out of

1. Atomic propositions: Capital letters $A, B, C, \dots$
2. Boolean connectives: $\wedge$, $\vee$, $\neg$, and $\rightarrow$
3. Parentheses:), (

Atomic Formulas

We use capital letters to represent **atomic formulas** or occasionally sentential letters: A, B, C, and so on (possibly with numeric subscripts).

When translating from English to formulas, you must provide a translation key (association of a letter with an atomic sentence).

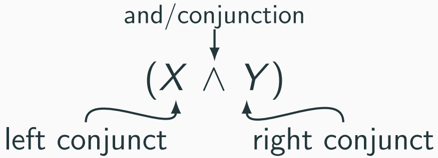
Variables

We use the letters (X, Y, Z) as variables that range over all formulas.

E.g., In algebra, you write '$y = x + 2$'. In this expression, 'x' is a variable that can be assigned any number.

Conjunction

If X and Y are formulas, then $(X \wedge Y)$ is a formula, called a **conjunction**.



Conjunction

Ann had coffee and Bob had tea.

[Ann had coffee]$_1$ and [Bob had tea]$_2$.

C	Ann had coffee.
T	Bob had tea.

C and T

$C \wedge T$.

Conjunction

Ann had coffee and Bob had tea.

[Ann had coffee]$_1$ and [Bob had tea]$_2$.

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Conjunction

Ann had coffee and Bob had tea.

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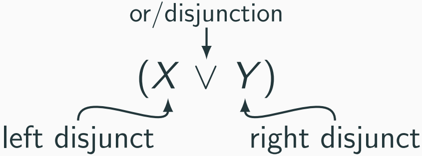
C	Ann had coffee.
T	Bob had tea.

C and T

$C \wedge T$

Disjunction

If X and Y are formulas, then $(X \vee Y)$ is a formula, called a **disjunction**.



Disjunction

Ann had coffee or Bob had tea.

[Ann had coffee]$_1$ or [Bob had tea]$_2$.

C	Ann had coffee.
T	Bob had tea.

C or T

$C \vee T$.

Disjunction

Ann had coffee or Bob had tea.

[Ann had coffee]$_1$ or [Bob had tea]$_2$.

C	Ann had coffee.
T	Bob had tea.

C or T

$C \wedge T$.

Disjunction

Ann had coffee or Bob had tea.

[Ann had coffee]$_1$ or [Bob had tea]$_2$.

C	Ann had coffee.
T	Bob had tea.

C or T

$C \vee T$.

Disjunction

Ann had coffee or Bob had tea.

[Ann had coffee]$_1$ or [Bob had tea]$_2$.

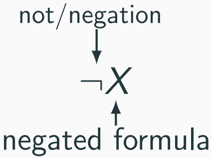
C	Ann had coffee.
T	Bob had tea.

C or T

$C \vee T$

Negation

If X is a formula, then $(\neg X)$ is a formula, called a **negation**.



Negation

Ann didn't have coffee.

It's not the case that Ann had coffee.

C	Ann had coffee.

It's not the case that [Ann had coffee]$_1$.

$\neg C$

Negation

Ann didn't have coffee.

It's not the case that Ann had coffee.

C	Ann had coffee.

It's not the case that [Ann had coffee]$_1$.

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Ann didn't have coffee.

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Negation

Ann didn't have coffee.

It's not the case that Ann had coffee.

C	Ann had coffee.

It's not the case that [Ann had coffee]$_1$.

$\neg C$

Conditional

If X and Y are formulas, then $(X \to Y)$ is a formula, called a **conditional** (sometimes this is called the **material conditional**).

conditional/implication/implies

$$(X \to Y)$$

antecedent

consequent

Conditional

If Ann had coffee, then Bob had tea.

[Ann had coffee]$_1$ and [Bob had tea]$_2$.

C	Ann had coffee.
T	Bob had tea.

C and T.

$C \wedge T$.

Conditional

If Ann had coffee, then Bob had tea.

If [Ann had coffee]$_1$, then [Bob had tea]$_2$.

C	Ann had coffee.
T	Bob had tea.

C and T.

$C \wedge T$.

Conditional

If Ann had coffee, then Bob had tea.

If [Ann had coffee]$_1$, then [Bob had tea]$_2$.

C	Ann had coffee.
T	Bob had tea.

If C, then T.

$C \to T$.

Conditional

If Ann had coffee, then Bob had tea.

If [Ann had coffee]$_1$, then [Bob had tea]$_2$.

C	Ann had coffee.
T	Bob had tea.

If C, then T.

$C \rightarrow T$

Formulas

1. Every atomic formula is a formula of sentential logic.
2. If X is a formula of sentential logic, then so is $\neg X$.
3. If X and Y are formulas of sentential logic, then so are each of the following:
   a. $(X \wedge Y)$
   b. $(X \vee Y)$
   c. $(X \rightarrow Y)$
4. An expression of sentential logic is a formula only if it can be constructed by one or more applications of the first three rules.

Why is $(\neg(A \wedge \neg B) \rightarrow (C \vee \neg D))$ a formula?

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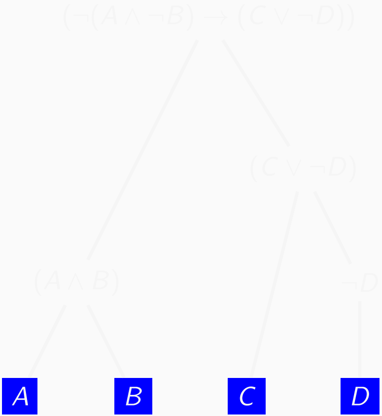

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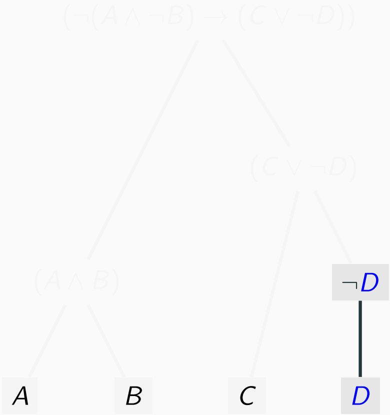

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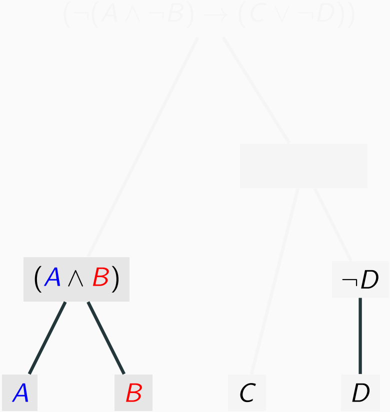

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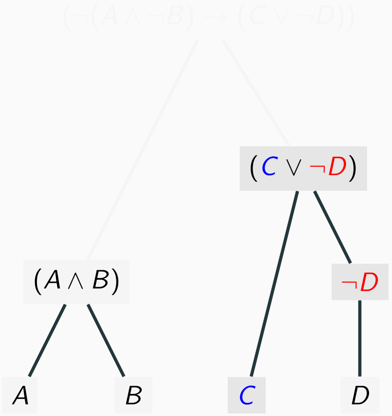

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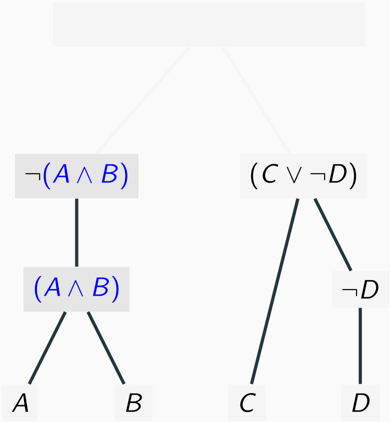

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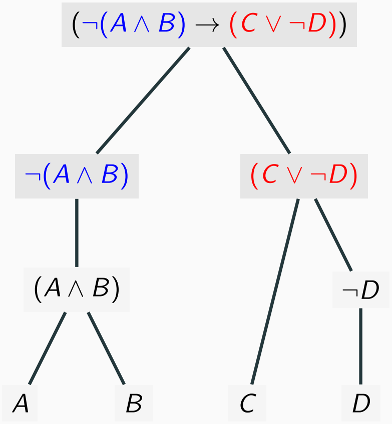

How do you find the syntax tree of a formula?

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What is the **main connective** of a formula?

1. If the first symbol in the formula is "$\neg$", then the first occurrence of "$\neg$" is the main connective.
2. Otherwise, the main connective is the first occurrence of the binary connective that is surrounded by the fewest number of parentheses.

$$(\neg(K \vee S) \rightarrow A)$$

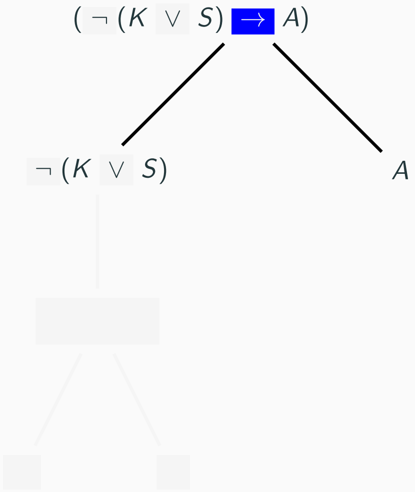
Syntax Tree

$(\neg(K \vee S) \rightarrow A)$

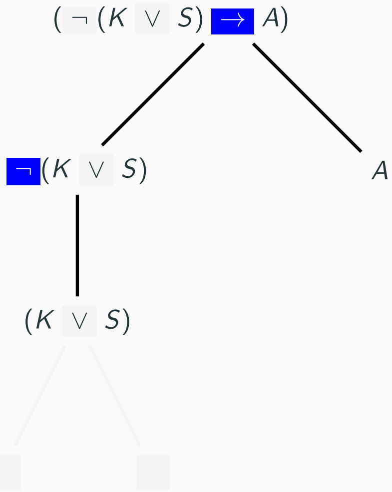
Syntax Tree

$(\neg(K \vee S) \rightarrow A)$

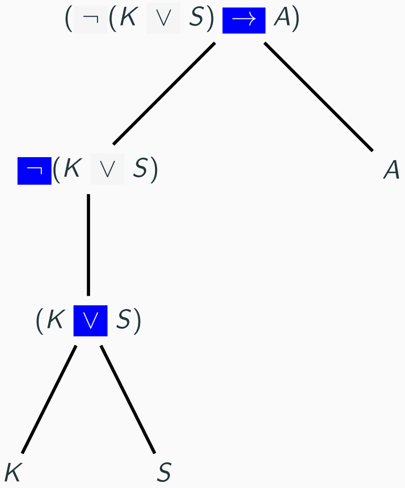
Syntax Tree



Syntax Tree

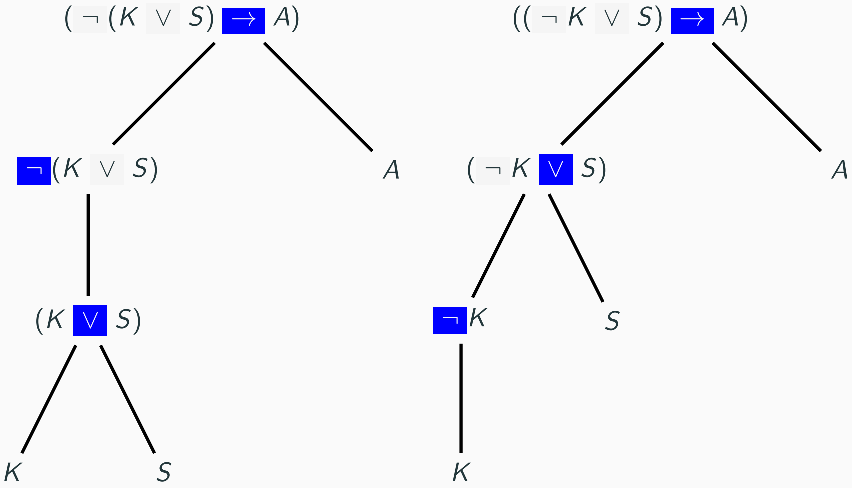


Syntax Tree



$(\neg(K \vee S) \rightarrow A)$

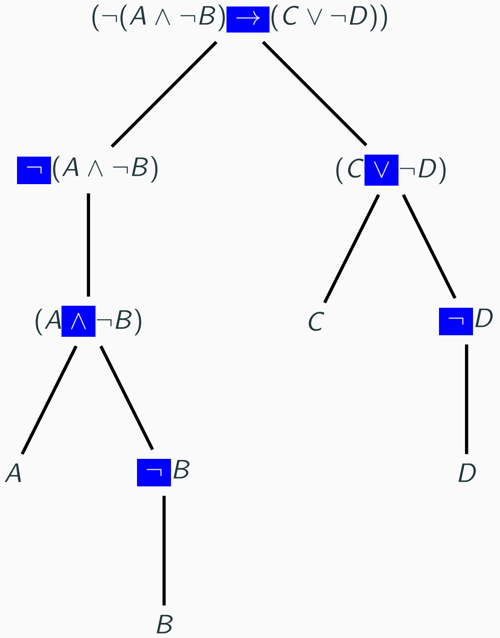
$((\neg K \vee S) \rightarrow A)$


(¬(K ∨ S) → A)
¬(K ∨ S)
A
(K ∨ S)
K
S
((¬K ∨ S) → A)
(¬K ∨ S)
A
¬K
S
K

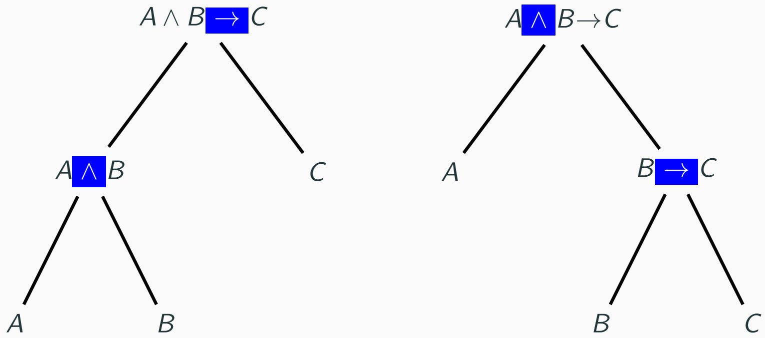
How do you find the syntax tree of a formula?

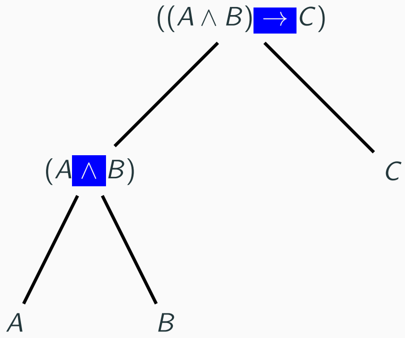
1. Write down the formula (adding parentheses if necessary).
2. Identify the main connective.
3. If it is a negation, write the negated formula below the current formula.
4. If it is a disjunction/conjunction/conditional, write the left disjunct/left conjunct/antecedent down left of the current formula and the right disjunct/right conjunct/consequent below right of the current formula.
5. Repeat steps 2-4 for for every formula that is not an atomic formula.
6. Draw lines connecting formula to the ones immediately below them.

Draw the syntax tree for $(\neg(A \wedge \neg B) \rightarrow (C \vee \neg D))$.


(¬(A ∧ ¬B) → (C ∨ ¬D))
¬(A ∧ ¬B)
(C ∨ ¬D)
(A ∧ ¬B)
C
¬D
A
¬B
D
B

$$A \wedge B \rightarrow C$$


A ∧ B → C
A ∧ B
C
A
B
A ∧ B → C
A
B → C
B
C


$((A \wedge B) \rightarrow C)$
$(A \wedge B)$
C
A
B

Procedure for Reinserting Omitted Parentheses

1. The outermost parentheses can be removed.

   For example, $((P \wedge R) \rightarrow Q)$ can be replaced with $(P \wedge R) \rightarrow Q$.

2. Group conjunctions and disjunctions before conditionals.

   For example, $(P \wedge Q) \rightarrow (R \vee S)$ is the same as $P \wedge Q \rightarrow R \vee S$.

3. Group conjunctions and disjunctions to the left before the ones on the right.

   For example, $(P \wedge Q) \wedge R$ is the same as $P \wedge Q \wedge R$.

4. Group conjunctions before disjunctions.

   For example, $P \vee (Q \wedge R)$ is the same as $P \vee Q \wedge R$.

Add Parentheses

$$A \wedge B \wedge C$$

$$A \wedge B \vee C$$

$$\neg A \rightarrow C$$

$$\neg A \wedge B \rightarrow C$$

$$\neg A \wedge B \vee C$$

Add Parentheses

$$((A \wedge B) \wedge C)$$

$$((A \wedge B) \vee C)$$

$$(\neg A \rightarrow C)$$

$$((\neg A \wedge B) \rightarrow C)$$

$$((\neg A \wedge B) \vee C)$$