Reasoning for Humans: Clear Thinking in an Uncertain World

PHIL 171

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Stochastic Truth Table

	P	Q	$\cdots$
p_1	T	T	
p_2	T	F	
p_3	F	T	
p_4	F	F	

A **Stochastic Truth Table** is a truth table where there is a number assigned to each row such that:

(1) each number is greater than or equal to 0; and

(2) and the sum of all the numbers assigned to the rows is 1

Given a stochastic truth table, how do you determine the probability of a formula X?

$Pr(X) =$ the sum of the probabilities assigned to rows that make X true.

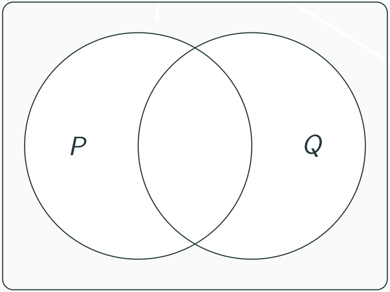
Conditional Probability

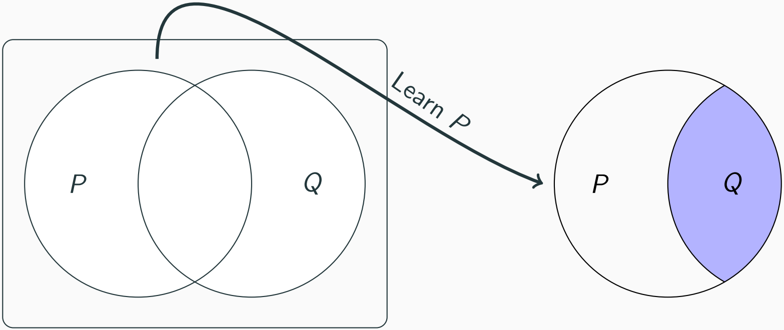
$Pr(X \mid Y)$: The probability of X **given that** Y.

Conditional probability is the concept of the probability of something *given* or *in the light of* some evidence or new information.

Example:

- the probability that the die lands 1, given that it lands odd, is 1/3
- the probability that it will rain tomorrow, given that there are dark clouds in the sky tomorrow morning, is high


P
Q



$Pr(Q \mid P)$

Ratio Formula

$$Pr(X \mid Y) = \frac{Pr(X \wedge Y)}{Pr(Y)}$$

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(If $Pr(Y) = 0$, then $Pr(X \mid Y)$ is undefined)

Kolmogrov Axioms

Given any stochastic truth table, for any formulas X and Y:

- $Pr(X) \geq 0$
- If X is a tautology, then $Pr(X) = 1$
- If X and Y are mutually exclusive, then $Pr(X \vee Y) = Pr(X) + Pr(Y)$

Laws of Probability

In any stochastic truth table, for all X, $Pr(\neg X) = 1 - Pr(X)$

In any stochastic truth table, for all X, if X is a contradiction, then $Pr(X) = 0$

In any stochastic truth table, for all X and Y, if $X \leftrightarrow Y$ is a tautology (i.e., X and Y are tautologically equivalent), then $Pr(X) = Pr(Y)$

Laws of Probability

In any stochastic truth table, for all X and Y,
$Pr(X) = Pr(X \wedge Y) + Pr(X \wedge \neg Y)$

In any stochastic truth table, for all X and Y,
$Pr(X \vee Y) = Pr(X) + Pr(Y) - Pr(X \wedge Y)$

Laws of Total Probability

In any stochastic truth table, for all X and Y,

$$Pr(X) = Pr(Y)Pr(X \mid Y) + Pr(\neg Y)Pr(X \mid \neg Y)$$

In any stochastic truth table, for all X and Y_1, Y_2 and Y_3, if

1. $Pr(Y_1 \vee Y_2 \vee Y_3) = 1$,
2. Y_1 and Y_2 are mutually exclusive, Y_2 and Y_3 are mutually exclusive, and Y_1 and Y_3 are mutually exclusive,

then

$$Pr(X) = Pr(Y_1)Pr(X \mid Y_1) + Pr(Y_2)Pr(X \mid Y_2) + Pr(Y_3)Pr(X \mid Y_3)$$

Conjunctions and Conditional Probability

In any stochastic truth table, for all X and Y,
$Pr(X \wedge Y) = Pr(X)Pr(Y \mid X)$

In any stochastic truth table, for all X, Y, and Z
$Pr(X \wedge Y \mid Z) = Pr(X \mid Z)Pr(Y \mid X \wedge Z)$

Conditionals in Stochastic Truth Tables

In any stochastic truth table, for all X and Y,
if $X \rightarrow Y$ is a tautology, then $Pr(X) \leq Pr(Y)$

Independence

Two formulas X and Y are **independent** given a stochastic truth table provided that $Pr(Y) = Pr(Y \mid X)$.

Equivalent ways of defining when two formulas X and Y are **independent** given a stochastic truth table:

- $Pr(X \wedge Y) = Pr(X)Pr(Y)$.
- $Pr(X) = Pr(X \mid Y)$.

Positive/Negative Relevance

X is **positively relevant** to Y (given a stochastic truth table) when $Pr(Y \mid X) > Pr(Y)$

X is **negatively relevant** to Y (given a stochastic truth table) when $Pr(Y \mid X) < Pr(Y)$

Evidential Support

X **evidentially supports** Y (given a stochastic truth table) when $Pr(Y \mid X) > 0.5$

Inductively Strong

The argument $P \Rightarrow C$ is inductively strong when:

1. P **evidentially supports** C: $Pr(C \mid P)$ is "high" (i.e., $Pr(C \mid P) > \frac{1}{2}$).
2. P is **positively relevant** to C: $Pr(C \mid P) > Pr(C)$
3. The argument is not deductively valid.

Posterior

Prior of H

$$Pr(H|E) = Pr(E|H)\frac{Pr(H)}{Pr(E)}$$

Likelihood

Prior of observing E

$$Pr(H \mid E) = Pr(E \mid H)\frac{Pr(H)}{Pr(E)}$$

$$Pr(H \mid E) = Pr(E \mid H)\frac{Pr(H)}{Pr(H)Pr(E \mid H) + Pr(\neg H)Pr(E \mid \neg H)}$$

Base-Rate Fallacy

A cab was involved in a hit and run accident at night. Two cab companies, the Green and the Blue, operate in the city. 85% of the cabs in the city are Green and 15% are Blue. A witness identified the cab as Blue. The court tested the reliability of the witness under the same circumstances that existed on the night of the accident and concluded that the witness correctly identified each one of the two colors 80% of the time and failed 20% of the time.

What is the probability that the cab involved in the accident was Blue rather than Green knowing that this witness identified it as Blue?

D. Kahneman. *Thinking, Fast and Slow*. Macmillan, 2011.

Base-Rate Fallacy

E	The witness identified the cab as blue.
H	A cab from the blue cab company was in the accident.
$\neg H$	A cab from the green cab company was in the accident.

What is $Pr(H \mid E)$?

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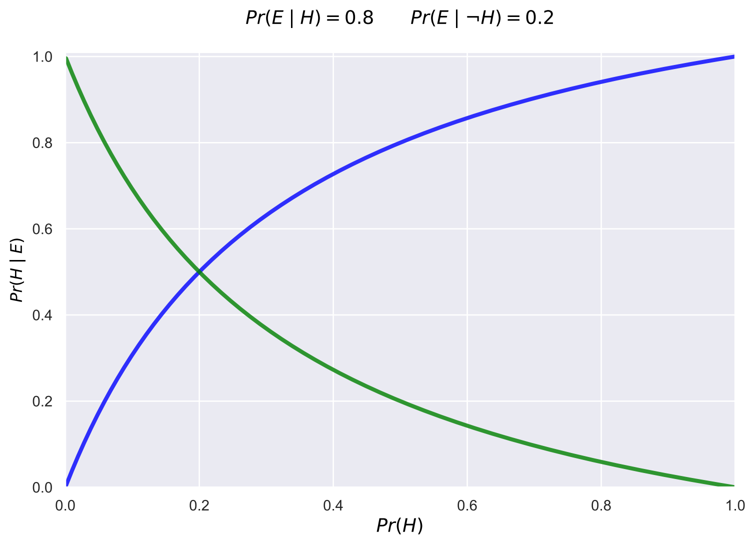
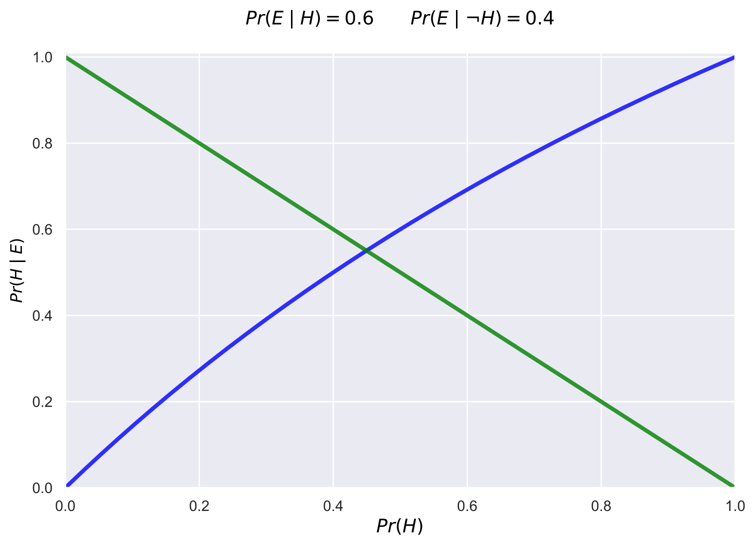
$$Pr(H \mid E) = Pr(E \mid H)\frac{Pr(H)}{Pr(E)} = 0.8\frac{0.15}{0.15*0.8+0.85*0.2} \approx 0.41$$

$$Pr(\neg H \mid E) = Pr(E \mid \neg H)\frac{Pr(\neg H)}{Pr(E)} = 0.2\frac{0.85}{0.15*0.8+0.85*0.2} \approx 0.59$$



$Pr(E \mid H) = 0.8 \quad Pr(E \mid \neg H) = 0.2$
1.0
0.8
0.6
0.4
0.2
0.0
$Pr(H \mid E)$
0.0
0.2
0.4
0.6
0.8
1.0
$Pr(H)$


Pr(E | H) = 0.6   Pr(E | ¬H) = 0.4
1.0
0.8
0.6
0.4
0.2
0.0
Pr(H | E)
0.0
0.2
0.4
0.6
0.8
1.0
Pr(H)

$$Pr(H \mid E) = Pr(E \mid H)\frac{Pr(H)}{Pr(E)}$$

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