Reasoning for Humans: Clear Thinking in an Uncertain World

PHIL 171

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Bayes Theorem

Posterior

Prior of H

$$Pr(H|E) = Pr(E|H)\frac{Pr(H)}{Pr(E)}$$

Likelihood

Prior of observing E

$$Pr(H \mid E) = Pr(E \mid H)\frac{Pr(H)}{Pr(E)}$$

$$Pr(H \mid E) = Pr(E \mid H)\frac{Pr(H)}{Pr(H)Pr(E \mid H) + Pr(\neg H)Pr(E \mid \neg H)}$$

Suppose you select one card from a standard deck of cards.

Q is "the card is a queen card" and F is "the card is a face card".

What is $Pr(Q \mid F)$?

	Q	F	$Q \wedge F$
$\frac{4}{52}$	T	T	T
0	T	F	F
$\frac{8}{52}$	F	T	F
$\frac{40}{52}$	F	F	F

$$Pr(Q \mid F) = \frac{Pr(Q \wedge F)}{Pr(F)} = \frac{\frac{4}{52}}{\frac{12}{52}} = \frac{4}{12} = \frac{1}{3}$$

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Note that:

$$Pr(F \mid Q) = 1$$

$$Pr(F) = \frac{12}{52}$$

$$Pr(Q) = \frac{4}{52}$$

$$Pr(Q \mid F) = Pr(F \mid Q)\frac{Pr(Q)}{Pr(F)} = 1 \times \frac{\frac{4}{52}}{\frac{12}{52}} = \frac{4}{12} = \frac{1}{3}$$

Three Prisoner's Problem

Three prisoners A, B and C have been tried for murder and their verdicts will told to them tomorrow morning.

They know only that one of them will be declared guilty and will be executed while the others will be set free. The identity of the condemned prisoner is revealed to the very reliable prison guard, but not to the prisoners themselves.

Prisoner A asks the guard "Please give this letter to one of my friends — to the one who is to be released. We both know that at least one of them will be released".

Three Prisoner's Problem

An hour later, A asks the guard "Can you tell me which of my friends you gave the letter to? It should give me no clue regarding my own status because, regardless of my fate, each of my friends had an equal chance of receiving my letter."

The guard told him that B received his letter.

Prisoner A then concluded that the probability that he will be released is $1/2$ (since the only ones without a verdict are A and C).

Three Prisoner's Problem

But, A thinks to himself:

> *Before I talked to the guard my chance of being executed was 1 in 3. Now that he told me B has been released, only C and I remain, so my chances of being executed have gone from 33.33% to 50%. What happened? I made certain not to ask for any information relevant to my own fate...*

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Explain what is wrong with A's reasoning.

A's reasoning

Consider the following events:

G_A: "Prisoner A will be declared guilty" (we have $Pr(G_A) = 1/3$)

I_B: "Prisoner B will be declared innocent" (we have $Pr(I_B) = 2/3$)

A's reasoning

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We have $Pr(I_B \mid G_A) = 1$: "If A is declared guilty then B will be declared innocent."

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Bayes Theorem:

$$Pr(G_A \mid I_B) =$$

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Bayes Theorem:

$$Pr(G_A \mid I_B) = Pr(I_B \mid G_A)\frac{Pr(G_A)}{Pr(I_B)} =$$

A's reasoning

Consider the following events:

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We have $Pr(I_B \mid G_A) = 1$: "If A is declared guilty then B will be declared innocent."

Bayes Theorem:

$$Pr(G_A \mid I_B) = Pr(I_B \mid G_A)\frac{Pr(G_A)}{Pr(I_B)} = 1 \cdot \frac{1/3}{2/3} = 1/2$$

A's reasoning, corrected

But, A did not receive the information that B *will be declared innocent*, but rather that *the guard said that B will be declared innocent.* So, A should have conditioned on the event:

I'_B: "The guard said that B will be declared innocent"

Given that $Pr(I'_B \mid G_A)$ is $1/2$ (given A is guilty, there is a 50-50 chance that the guard could have given the letter to B or C).

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But, A did not receive the information that B *will be declared innocent*, but rather that *the guard said that B will be declared innocent.* So, A should have conditioned on the event:

I'_B: "The guard said that B will be declared innocent"

Given that $Pr(I'_B \mid G_A)$ is $1/2$ (given A is guilty, there is a 50-50 chance that the guard could have given the letter to B or C). This gives us the following correct calculation:

$$Pr(G_A \mid I'_B) = Pr(I'_B \mid G_A)\frac{Pr(G_A)}{Pr(I'_B)} = 1/2 \cdot \frac{1/3}{1/2} = 1/3$$

Monty Hall Dilemma

Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say number 1, and the host, who knows what's behind the doors, opens another door, say number 3, which has a goat. He says to you, "Do you want to pick door number 2?" Is it to your advantage to switch your choice of doors?

Monty Hall (1)

H_1: The car is behind door 1

H_2: The car is behind door 2

H_3: The car is behind door 3

Monty Hall (2)

Reasoning 1: E: The car is not behind door 3 ($\neg H_3 \leftrightarrow H_1 \vee H_2$)

Monty Hall (2)

Reasoning 1: E: The car is not behind door 3 ($\neg H_3 \leftrightarrow H_1 \vee H_2$)

$$Pr(H_1 \mid E) \quad = \quad Pr(E \mid H_1)\frac{Pr(H_1)}{Pr(E)}$$

Monty Hall (2)

Reasoning 1: E: The car is not behind door 3 ($\neg H_3 \leftrightarrow H_1 \vee H_2$)

$$\begin{aligned} Pr(H_1 \mid E) &= Pr(E \mid H_1)\frac{Pr(H_1)}{Pr(E)} \\ &= \frac{Pr(E \mid H_1)Pr(H_1)}{Pr(E \mid H_1)Pr(H_1)+Pr(E \mid H_2)Pr(H_2)+Pr(E \mid H_3)Pr(H_3)} \end{aligned}$$

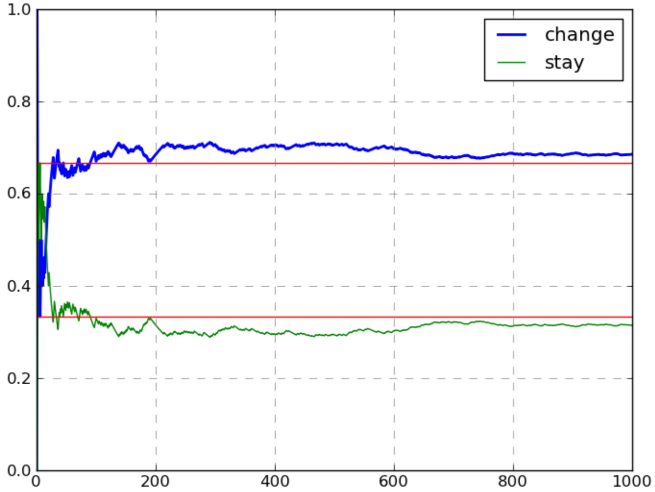
Monty Hall (2)

Reasoning 1: E: The car is not behind door 3 ($\neg H_3 \leftrightarrow H_1 \vee H_2$)

$$
\begin{aligned}
Pr(H_1 \mid E) &= Pr(E \mid H_1)\frac{Pr(H_1)}{Pr(E)} \\
&= \frac{Pr(E \mid H_1)Pr(H_1)}{Pr(E \mid H_1)Pr(H_1)+Pr(E \mid H_2)Pr(H_2)+Pr(E \mid H_3)Pr(H_3)} \\
&= 1 \cdot \frac{\frac{1}{3}}{1\cdot\frac{1}{3}+1\cdot\frac{1}{3}+0\cdot\frac{1}{3}} \\
&= 1 \cdot \frac{\frac{1}{3}}{\frac{2}{3}} \\
&= \frac{1}{2}
\end{aligned}
$$

Similarly for $Pr(H_2 \mid E)$, so do not switch.

Monty Hall: Reasoning 1 vs. Reasoning 2



Monty Hall (3)

Reasoning 2: F: Monty opened door number 3

$$
\begin{aligned}
Pr(H_2 \mid F) &= Pr(F \mid H_2)\frac{Pr(H_2)}{Pr(F)} \\
&= \frac{Pr(F \mid H_2)Pr(H_2)}{Pr(F \mid H_1)Pr(H_1)+Pr(F \mid H_2)Pr(H_2)+Pr(F \mid H_3)Pr(H_3)} \\
&= 1 \cdot \frac{\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+1\cdot\frac{1}{3}+0\cdot\frac{1}{3}} \\
&= 1 \cdot \frac{\frac{1}{3}}{\frac{1}{2}} \\
&= \frac{2}{3}
\end{aligned}
$$

So, $Pr(H_1 \mid F) = \frac{1}{3}$ and $Pr(H_2 \mid F) = \frac{2}{3}$, so you should switch

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Reasoning 2: F: Monty opened door number 3

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Pr(H_2 \mid F) &= Pr(F \mid H_2)\frac{Pr(H_2)}{Pr(F)} \\
&= \frac{Pr(F \mid H_2)Pr(H_2)}{Pr(F \mid H_1)Pr(H_1)+Pr(F \mid H_2)Pr(H_2)+Pr(F \mid H_3)Pr(H_3)} \\
&= 1 \cdot \frac{\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+1\cdot\frac{1}{3}+0\cdot\frac{1}{3}} \\
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So, $Pr(H_1 \mid F) = \frac{1}{3}$ and $Pr(H_2 \mid F) = \frac{2}{3}$, so you should switch